PROBLEMS OF THE 54th—26th INTERNATIONAL—RUDOLF ORTVAY PROBLEM SOLVING CONTEST IN PHYSICS 2—12 February 2024

1. The attached photo was taken close to the Piszkéstető Observatory before sunset. Several mountain peaks can be identified, like Ágasvár, Tepke, Nagy-hegy, Csóványos, Szandavár, Szitnya and so on.

Determine the radius of the Earth and the altitude of the camera using the photo as precisely as possible, and give the uncertainties of your results.

You can make use of the altitude (above sea level) of the visible mountains in the picture, as well as their distances from Piszkéstető and from each other (you can read these data off from your favourite online map), since these quantities could be measured, in principle, with sufficient effort using basic methods (with meter-sticks and barometers, for example), which are, of course, not part of this exercise.

The original high resolution version of the photo can be downloaded from the website:

https://ortvay.elte.hu/2024/piszkes.jpg

(Gábor Veres and Vázsony Varga)

2. Inside the Earth a circular railway is built. The railway is in a completely general position inside the Earth, which is considered to be spherical and homogeneous in mass distribution. In the tunnel, the train moves without friction, under the sole effect of gravity. The effects of the Earth's rotation can be neglected.

a) Derive the equation of motion of a point-like train.

b) Consider the special case where the track is located in the plane of one of the Earth's great circles. The railway wagon with a given initial velocity starts on the track at the point which is α) furthest from β) nearest to the center of the Earth.

Calculate the period of motion in both cases as a function of the parameters of the trajectory and the initial speed.

(József Cserti)

3. The two-person swing shown in the photo appeared on a few playgrounds. The parent drives the device in the left seat as if swinging alone. The small child is sitting in the seat on the right. Some of the children like this swing because it offers a different experience than the usual ones.

Let us make a simplified point mechanical model of the system, with data close to reality. Assume that the parent's driving acts as a periodic torque.

Simulate numerically the motion of the child's swing.

Is it possible that this movement is chaotic in the realistic swing range of no more than 90 degrees?

(Tamás Tél, grandpa)

4. The all-sky camera of Piszkéstető Observatory (https://www.idokep.hu/webkamera/piszkesteto_allsky) takes one picture every 20 seconds, which can be downloaded one by one (https://ccdsh.konkoly.hu/share/allsky/).

Measure the apparent path of a satellite pass of your choice from this year with this camera, and fit an appropriate function to the data to determine the satellite's altitude above the surface (and its uncertainty) as precisely as possible. Discuss possible systematic errors too.

In case of really accurate measurements, knowing the surface gravitational acceleration, the Earth's radius could be determined – what kind of satellite would have to be chosen for this purpose?

In addition to the data measured from the images, you can use the Earth's data and celestial coordinates of stars. You can consider the Earth as spherically symmetric and the orbit as circular, but discuss the uncertainties originating from these assumptions. Try to reach a relative statistical uncertainty in the altitude above the surface of less than one percent.

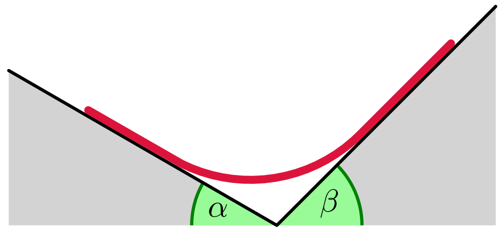
Check (e.g. on Heavens-Above, https://www.heavens-above.com/AllSats.aspx) if the measured object is indeed a satellite and if the eccentricity of its orbit is sufficiently small.

To calibrate the conversion between pixel coordinates and horizontal coordinates, we recommend using the web (https://stellarium-web.org/) or desktop (http://stellarium.org/) version of Stellarium – make sure to use the correct time, time zone and location.

In addition to presenting your work and figures, please indicate exactly which pass you have used (e.g. Heavens-Above link, or satellite marked on a screenshot).

(Vázsony Varga and Gábor Veres)

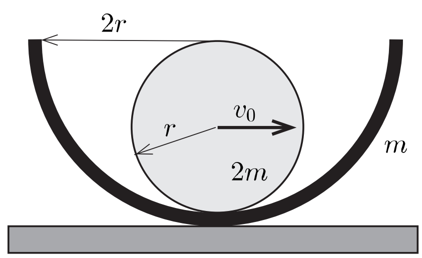
5. A homogeneous cord rests on a pair of connected, rigid slopes with adjustable inclination angles, as shown in the figure. The coefficient of friction between the slopes and the cord is μ.



How should we choose the inclination angles α and β, and how should we deposit the cord to make the length of its hanging middle section maximal?

(Ákos Gombkötő and Róbert Németh)

6. A thin-walled half-tube of mass m and radius $2r$ can roll without slipping on a horizontal surface. A cylinder of mass $2m$, radius r, with homogeneous mass density, can roll without slipping inside the half-tube. At instant $t = 0$, the half-tube is at rest, while the axis of the cylinder moves with a small horizontal speed v_0.



Calculate the angular displacement of the half-tube as a function of time.

(Jenő Gelencsér and Péter Gnädig)

7. Alice and Bob are fascinated by the intellectual heritage of Eratosthenes, and decide to repeat his famous measurement of the circumference of the Earth. In order to do that, they measure very very precisely the length of the shadow of a vertical stick at the same moment; Alice in a village named Nak in Hungary, and Bob at the ELTE Lágymányos campus in Budapest, on a day when the altitude of the Sun at solar noon in Budapest was precisely 45 degrees. They calculate the altitute of the Sun from the shadow lengths at their location, at the time of the measurement, then they take the difference of these two angles. They read off the distance of their two locations from the map, they divide that by the above angular difference, then multiply it by 360 degrees, thus obtaining the circumference of the Earth. They are surprised that their result is infinity, i.e. that the Earth is flat!

   They are worried about this result, thus repeat their measurement a little later on the same day. Now their result is 80 thousand kilometers! They suspect that they may have made a mistake of a factor of 2 in their calculations, but after checking everything they conclude that the calculation is correct.

   Thus after a little while they repeat their measurement again. This third time, they get 40 thousand kilometers as a result!

   Question: at what time did they perform each of their three measurements? (The solar noon in Budapest on that day was at 12:39 pm.)

(Gábor Veres)

8. A rope of length L and mass m lies on a horizontal surface such that its shape is a straight line. One end of the rope is lifted *slowly* along the vertical line. What is the work required to lift the rope until the other end loses contact with the surface? The coefficients of static and kinetic friction between the rope and the surface are both equal to $\mu = 0.3$.

(Máté Vigh)

9. A circular, very heavy ring is placed horizontally. One of its diameters is made of a thin, rigid wire of length $10a$. A drilled ball of mass $4m$ is placed in the middle of this wire, and a spring with a rest length a and stiffness k, strung on the wire, is attached to both sides of the central ball. We place two balls of mass $3m$ at the outer end of both springs. These balls are connected to the outer end point of the diameter by a spring of rest length $4a$, but also with stiffness k. The outer end of these springs is fixed to the end points of the diameter. The balls can move freely along the wire without friction, but are always connected by the springs. The system is rotated with an angular velocity Ω around an axis perpendicular to the plane of the ring. The effect of the motion of the balls on the rotation of the ring can be neglected.

   Hint: it is advisable to introduce the fundamental frequency $\omega_0 = \sqrt{k/m}$ and the dimensionless parameter $p = 12\,\Omega^2/\omega_0^2$, and to answer the following to questions in those terms.

   a) Where are the stationary positions of each ball (i.e., those points at which the balls remain at rest as seen from the system co-rotating with the wheel)?

   b) Calculate the frequencies of the small vibrations around the stationary positions and determine the displacement patterns of the corresponding normal modes. What is the upper limit on the amplitudes of these normal modes?

   c) Let's examine the stability of the stationary positions as a function of the parameter p.

   d) Does the system feature zero-frequency normal modes and symmetry-breaking stationary states?

   e) Let the value of the parameter p now be $p = 23/12$. In the time period $t < 0$, the system rotates with an angular velocity Ω, and the balls oscillate with the lowest frequency. At time $t = 0$, the vibration is just at the zero displacement relative to the stationary positions. At this moment, the movement of the middle ball is momentarily stopped, and then the ball is released.
   Determine the movement of each ball as a function of time.
   Does this initial state at $t = 0$ ever recover along the diameter? If so, when is the earliest?

(Gyula Dávid)

10. Astronomers on Goomy Coast have observed six consecutive positions of a planet moving in the central force field of a strange distant star. Unfortunately, the enthusiastic researchers forgot to record the time of the sightings. Later, the system was obscured by a cosmic cloud, so the observation could not be continued.
    The detected positions of the planet fit exactly on a circle, the center of which, however, did not coincide with the center of the central force field. The radius b of the apparent circular trajectory was smaller than the distance c of its center from the center of the force field.

    Derive the formula of the central potential $V(r)$. Let's outline the earlier and later stages of the orbit of the orbiting planet.

    (Work strictly within the framework of classical mechanics, neglect all effects of special relativity.)

(based on a publication by A. Tudde Mynek, Port Goomy Daily Shocking Science)

11. Super-precise Galilei performs free-fall experiments, dropping balls from windows on different floors of the Burj Khalifa tower.

    What is the accuracy needed on the falling time of the dropped bodies if the purpose of the measurement is to determine the radius of Earth? What formula and graph should one use to evaluate your measurement (for some reason he couldn't get hold of a computer)?

    Assume either that a) there is no atmosphere, b) or the air exerts a drag force proportional to the speed of the falling body.

    The Earth can be regarded as a regular sphere, the effect of its rotation can be neglected.

(Gyula Dávid)

12. Let's reconsider the famous question regarding the shape of a frictionless incline, aiming to determine the optimal form for a slope on which a point of mass, launched from a given height under the influence of vertical forces, reaches the initial height in the shortest time possible.

    a) Seek the shape of the incline in the form $y = y(x)$! First, let us consider an unconventional potential:

$$V(y) = -\frac{a}{y^2}.$$

    Initiate the particle from the point $x_0 = 0$ at a positive height y_0, with finite initial velocity, ensuring the total energy of the particle is zero. Utilize the known methods of variational calculus to find stationary paths (those with zero first variation) where the endpoint $y_1 = y(x_1) = y_0$ is reached. Compute the total times as well.

    Examine the stability of the obtained solutions using the functional's second variation. Discuss qualitatively the potential solutions as a function of x_1, with fixed $y_1 = y_0$. Is it possible to find a trajectory that yields a shorter travel time than the stationary variational solution?

    b) Now, start with well-known gravitational potential:

$$V(y) = mgy.$$

    Look for paths where the particle, starting from ($x_0 = 0$, $y_0 = 0$), reaches the point (x_1, y_1) as quickly as possible, but this time without fixing the value of y_1. Under these boundary conditions, what solutions do we find? What are the corresponding total times?

    Examine the stability of the solutions as well. What interesting features do we observe?

    Hint: Attempt to calculate the change in the total time after adding a small variation $\varepsilon\eta(x)$ chosen by us, both in first and second order of ε.

(Kornél Kapás)

13. Consider a rigid body of irregular shape (e.g. an asteroid) in the inhomogeneous gravitational field of the neighbouring celestial bodies. Assume that the size of the body is much smaller than the characteristic length describing the inhomogeneity of the surrounding gravitational field.
    Calculate the torque of the gravitational forces acting on the body.

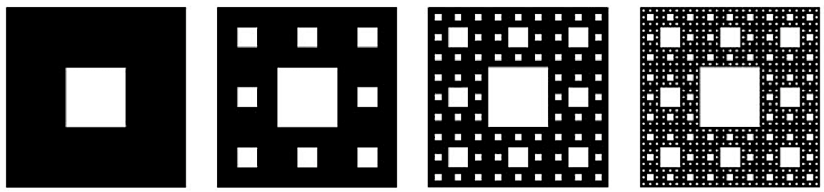
    Can you locate the body in a given point of the space in a sufficient position thus the torque of the tidal forces vanishes? How many different orientations can you find in the general case? Study the special degenerate cases as well.

(Gyula Dávid)

14. A polarizable molecule of fixed position is bombarded by a homogeneous ion beam of low particle density and velocity v. The distant electromagnetic interaction between the ion and molecule can be modeled by a classical, effective potential $V(r) = -\alpha/r^4$. Should they come close enough to each other, a chemical reaction occurs. Calculate the total cross section of the reaction as a function of ion energy.

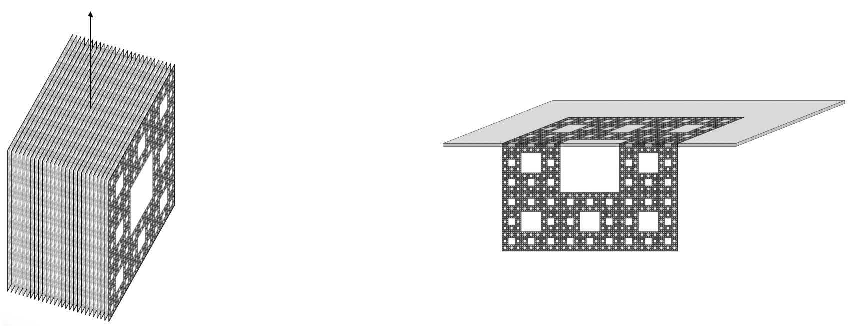
(Ákos Gombkötő)

15. A stack of sixth phase Sierpiński carpets are hung in Sierpiński's carpet factory, parallel to each other. A Sierpiński carpet in phase 0 is just a square carpet. In phase 1, a square area, the middle 1/9, is cut out of it.



The carpet manufacturing process, up to phase six, can be seen here: https://hu.wikipedia.org/wiki/Fájl:Animated_Sierpinski_carpet.gif.

a) From the carpets hung close to each other, one is pulled out very slowly vertically upwards, according to the illustration below. Those next to it do not move, but they act on the extracted carpet with a frictional force which is proportional to the contact surface area. How much work (mechanical energy) is needed to lift the carpet, if the side length of the carpet is L, its mass is M, and the frictional force coefficient is $\lambda\,[N/\mathrm{m}^2]$?



b) These carpets are very flexible. How far does the same sixth-phase carpet have to be pulled off the side of a table (see right panel of figure) so that the hanging part already pulls the rest with it? The adhesion friction coefficient between the carpet and the table is $\mu = 0.375$.

(Merse Előd Gáspár)

16. Consider a one-dimensional ball-spring system consisting of N balls each of mass m and one ball of mass M, where $M \gg m$. The spring constant is k.

Consider the motion of the mass M in the thermodynamic limit (i.e. when $N \to \infty$) on a timescale for which $t \gg \omega^{-1} = \sqrt{m/k}$. In this case, the motion of the mass M can be described by the Langevin equation. Based on this, determine the effective friction coefficient γ of the mass M. How does the ratio of the relaxation time γ^{-1} to ω^{-1} depend on the ratio of the masses?
Write a simple program that numerically solves the problem for different values of N and interpret the result.

(Máté Vass)

17. A parallel beam of light is incident on a glass cylinder of elliptical cross-section with refractive index n that makes an angle δ with the major axis of the ellipse.

a) Calculate and plot the caustics of the refracted rays inside the glass block.

b) Find the part of the incident light beam that is involved in the formation of the caustics inside the glass. Give the boundaries of this part of the beam.

Draw figures and investigate the phenomenon for different values of the refractive index n, the eccentricity of the ellipse ε and the angle δ.

Hint: use the parametric equation of the ellipse $x = a\cos t$, $y = b\sin t$, calculate the points of the caustics and find the boundaries of the incident beam in question b) in terms of the parameter t.

(Gyula Dávid and József Cserti)

18. We measured the power output of a solar panel in Budapest on a perfectly clear day (no clouds) on the 19th of June (see table below).

Question: What was the fraction of solar energy that is absorbed or reflected by the atmosphere at solar noon; at 8:00 in the morning; and at 19:00 in the evening?

In order to solve this problem, consider the following simple model: The solar panel was warmer than the air due to the absorbed solar radiation, and had a temperature of $T_{\mathrm{p}} = T_{\mathrm{a}} + K \cdot P$, where K is a constant, P is the power output of the solar panel and T_{a} is the temperature of the air, that was also measured separately and can be approximated by the equation

$$T_{\mathrm{a}}(t) = 29.1 - 0.16 \cdot (t - 14.6)^2 \, [^\circ\mathrm{C}]$$

where t is the local time (measured in hours).

We also know that the efficiency of the solar panel (to convert solar energy to electricity) decreases with increasing temperature by $0.35\,\%/^\circ\mathrm{C}$, thus we need to correct for that. The solar panel is tilted by 4 degrees with respect to the horizontal plane (i.e. almost horizontal), and oriented at 184 degrees (i.e. almost south, but 4 degrees off to the west).

The position of the Sun on this day, as a function of time, can be found on popular web-based calculators, which can be used to solve this problem. Consider that the intensity of solar radiation decreases exponentially with the thickness of the air it crosses. The constant K shall be chosen (fitted) in a way that this is satisfied.

A hint: consider all quantities as a function of the Sun's altitude!

The measured power output (in kW) is below, for every 10 minutes, starting at 7:20 in the morning (we always use standard summer time):

	0,81	0,90	0,98	1,07	1,16	1,23	1,32	1,40	1,47	1,55	1,63	
1,70	1,76	1,83	1,89	1,94	2,00	2,04	2,10	2,15	2,17	2,23	2,28	
2,29	2,31	2,34	2,36	2,35	2,38	2,41	2,40	2,40	2,43	2,39	2,42	
2,44	2,41	2,41	2,37	2,34	2,33	2,30	2,28	2,24	2,21	2,17	2,12	
2,07	2,02	1,98	1,91	1,87	1,81	1,75	1,68	1,61	1,54	1,46	1,39	
1,32	1,24	1,16	1,07	0,99	0,90	0,82	0,73	0,65	0,55	0,47	0,39	0,31

(Gábor Veres)

19. Let us consider a cube-shaped dielectric body with cubic crystal symmetry, placed in a constant $\boldsymbol{E}$ external electric field, which is thermally insulated and has a small electric susceptibility ($\chi \ll 1$). The value of the external pressure is p. How does the volume change of the cube depend on the electric field? How does the initial temperature T of the cube change? What if the cube is electrically conductive? Compare the result with the isotropic case!

(Máté Vass)

20. A space station hovers in space without acceleration, far away from any gravitational centers. An expedition ship leaves the station, and makes a round trip. The trajectory of the ship is an exact circle in the inertial system of the station. The ship first accelerates, and after reaching the farthest point of the trip, it decelerates, arriving with zero speed at the station. The magnitude of the acceleration is always g, i.e. the standard gravitational acceleration on the surface of Earth, as observed by the ship's crew.

How much time does the round trip take, and how much do the crew members age if the radius of the trip is 10 kilometers? And if the radius is 10 light-years?

Write down the equations of motion, and solve the problem with an accurate numerical method.

(Zsolt Bihary)

21. In a two-dimensional Minkowski space a spacecraft moves with a constant self-acceleration g in the direction of the x-axis, where g is the same as the gravitational acceleration measured on the Earth's surface. (This means that astronauts constantly feel their weight like on Earth during the motion.) The Minkowski space is defined by coordinates (t, x).

Let's re-define these coordinates in the largest possible range of space-time around the spaceship so that the constant-time surfaces coincide with the 'now' surfaces defined in the momentary inertial system of the astronauts. This time coordinate parameter T should be the same as the spaceship's proper time, and the constant spatial coordinate surfaces should connect the points that are equidistant from the spacecraft in the momentary inertial system of the spacecraft. Choose the spatial coordinate X in such a way that the square of the line element has a conformal Minkowski shape.

a) Write down conversion rules between coordinates (t, x) and (T, X), in both directions.

The coordinate system (T, X) can be considered as a space-time with a special Riemann-metric and can be treated with the methods of general relativity.

b) Derive and solve the scalar wave equation in the coordinate system (T, X).

c) Expand the scalar field in terms of plane waves in both coordinate systems. Give the relationship between the expansion coefficients valid in the two systems.

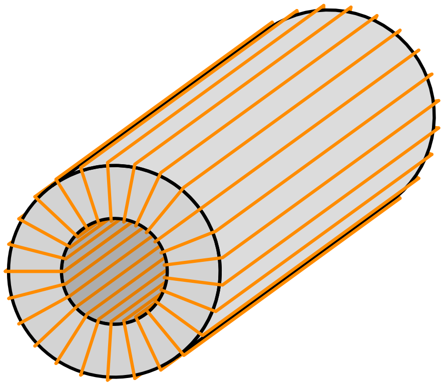
(Gyula Dávid)

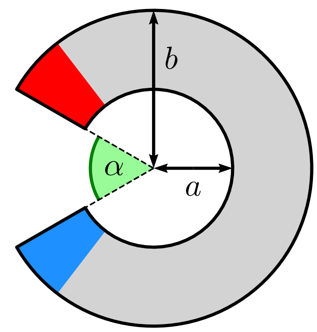
22. Consider two conducting earthed (hollow) spherical surfaces in space, centered at points given by the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, with radii of R_1 and R_2, respectively. Find those points in space where one can place a point charge such that exactly half of the electric field lines from the point charge arrive on each sphere in the emerging static electric field.

There is no restriction on the values of the parameters. We even allow either (or both) spheres to have 'infinite radii', i.e., to degenerate into planes.

(Ábel Tóth)

23. An initially unmagnetized cylindrical ring of ferromagnetic material has internal and external radii of a and b, respectively, while its extension along the axis is much larger. The body is winded uniformly with a current-carrying wire according to the figure on the left, making it magnetized. We may assume that the local residual magnetization is proportional to the local magnetic field due to the current.





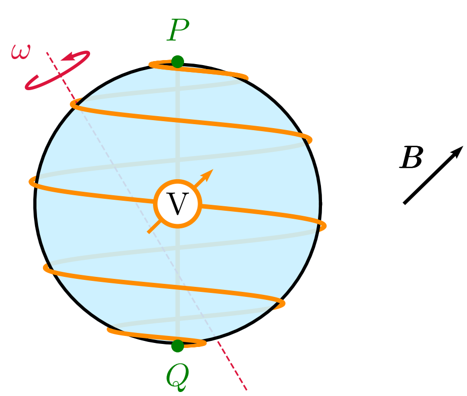
After this, a section of the ring, under an angle α as measured from the symmetry axis, is removed without modifying the local magnetization. The resulting system, shown in the figure on the right, may be considered as an approximate model of a horseshoe magnet if $\alpha \approx \pi$. Find an analytical expression for the vector potential associated with the static magnetic field in this special case.

(Ákos Gombkötő)

24. A thin wire loop of arbitrary shape is rotated around some axis with a stationary angular velocity $\boldsymbol{\omega}$ in a homogeneous magnetic field of flux density $\boldsymbol{B}$.

a) What is the induced voltage in the loop? Express it as a function of time in the simplest possible form.

b) Calculate the root mean square voltage as well. How should we adjust the orientation of the loop, the direction of the angular velocity and the magnetic field to make this minimal or maximal?



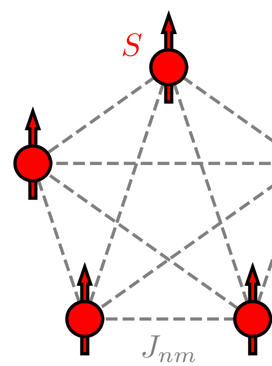
c) Consider the example depicted in the figure: N turns of wire wound up on the surface of a sphere of radius R between two opposite points P and Q, closed by a straight segment connecting the poles. The pitch of the coil is uniform with respect to the polar angle of the PQ axis. Examine the questions of exercise b) in this specific case. Discuss the result for different values of the number of turns.

(Róbert Németh)

25. The atoms of a strange ferromagnetic molecule are arranged in the vertices of a regular N-gon whose circumradius is R. Each atom possesses an internal angular momentum (spin) of magnitude S, and of continuously varying direction. Between them, there is a Heisenberg-type interaction, namely, for spins $\boldsymbol{S}_n$ and $\boldsymbol{S}_m$ with relative position $\boldsymbol{r}_{nm}$, the interaction energy is

$$U(\boldsymbol{S}_n, \boldsymbol{S}_m) = -2J_{nm}\left(\boldsymbol{S}_n \cdot \boldsymbol{S}_m\right), \quad \text{where} \quad J_{nm} = \frac{J_0 R^2}{|\boldsymbol{r}_{nm}|^2} > 0,$$

that is, the exchange constant is proportional to the inverse square distance.



Examine the classical dynamics of the molecule following the steps below:

a) Show that every configuration with aligned spins is an equilibrium state of the system. What can we tell about the stability of this arrangement?

b) Describe the motion of the system when the initially aligned spins are slightly diverted from their equilibrium orientations. Determine the dispersion relation corresponding to particular solutions.

c) Compute the general solution of the linearised equations of motion that can be fitted to any initial state.

(Róbert Németh)

26. Consider again the N-atom ferromagnetic molecule discussed in Problem 25, now taking a quantum mechanical approach. The Hamiltonian of the system is

$$\hat{H} = -\sum_{\substack{n=1 \\ n\neq m}}^{N}\sum_{m=1}^{N} J_{nm}\left(\hat{S}_n^x\hat{S}_m^x + \hat{S}_n^y\hat{S}_m^y + \hat{S}_n^z\hat{S}_m^z\right),$$

where $\hat{S}_n^x$, $\hat{S}_n^y$, $\hat{S}_n^z$ are components of the spin operator, and J_{nm} is the exchange constant proportional to the inverse square distance given before.

a) Show that every state where all spin projections onto a specified axis take a value of S with probability one is a ground state of the system. What is the corresponding eigenenergy?

b) Imagine that we excite the nth atom, that is, we modify its ground state specific definite spin projection by a value of $\hbar$. Then the molecule is left alone until, after some time t, we measure the same spin projection of the mth atom. What is the probability of finding it in an excited state? What happens in the specific case of $t = \pi/J_0 S$?

Hint: It is recommended to solve the classical case first (see Problem 25).

(Róbert Németh)

27. We learn in quantum mechanics, that sometimes the spectrum of a Hamiltonian can be found by purely algebraic methods. The main example for this is the harmonic oscillator: there we can compute the spectrum using the algebra of the creation/annihilation operators. However, this is not the only such example in theoretical physics. Let us now consider a different algebra, and compute the spectrum of the Hamiltonian given below.

Let $\hat{h}_j$, $j = 1, \ldots, N$ be Hermitian operators, $N \geq 1$, with the properties

$$\hat{h}_j^2 = 1, \qquad \operatorname{Tr} \hat{h}_j = 0$$

and satisfying the commutation relations

$$\begin{aligned} &\hat{h}_j \hat{h}_{j+1} = -\hat{h}_{j+1} \hat{h}_j \\ &[\hat{h}_j, \hat{h}_k] = 0, \quad \text{if} \quad |j-k| > 1. \end{aligned}$$

Define the Hamiltonian as

$$\hat{H} = \sum_{j=1}^{N} \hat{h}_j.$$

Determine the spectrum of this operator. The question for a generic N is highly complex, but certain cases detailed below can be computed relatively easily, so let us solve those problems.

a) Find representations of the algebra for small N. Which finite matrices can represent the operators $\hat{h}_j$ for $N = 1, 2, 3, 4$? Hint: one could make use of the Pauli matrices and their tensor products.

b) After finding the representations, calculate the spectrum of the Hamiltonian H for these small values of N.

c) Now determine the spectrum for $N = 1, 2, 3, 4$, independently from specific representations. We can assume that all operators act on finite dimensional spaces. Prove that the set of the eigenvalues does not depend on the representation, but the multiplicities of the eigenvalues can vary.

(Balázs Pozsgay)

28. We would like to conduct the usual EPR experiment with spins. Two particles originating from the decay of a spin-0 particle fly out, one in the left, and one in the right direction. If we measure spin 'up' along the axis z on the left side, then we get spin 'down' on the right side, and vice versa. The correlation between the right side and left side measurements is 100 %.

However, our experiment is disturbed by the Evil Permutators on both side of the experiment, and we can only access the experimental data through them.

The Evil Permutators' activity is the following (and they operate independently): they take 5—20 consecutive experimental results and make a permutation on them in an evil and random way. They only send us the resulting data only after all the measurements are done. They decide in a random way how many measurements they permute at a time as well. This way the left side measurements will not match the right side ones.

Do we still have a chance to find out if the left and right side data was correlated originally? If yes, how many measurements should we plan for a given accuracy?

(Győző Egri)

29. Generalize the two-dimensional massless Dirac equation. The Hamiltonian of a particle with electric charge Q moving in the (x, y) plane is given by

$$\hat{H} = \alpha\, \hat{\boldsymbol{p}}^2\, \hat{I} + \beta \left(\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{S}}\right)^2,$$

where α and β are constant parameters, $\hat{\boldsymbol{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$ is the momentum operator of the particle, $\hat{\boldsymbol{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ is the spin operator of an arbitrary spin S in units of $\hbar$ (integer or half integer, i.e., $S = 1/2,\ 1,\ 3/2,\ 2,\ \ldots$), and finally $\hat{I}$ is the $(2S+1)$-dimensional unit matrix. Since the motion is on the (x, y) plane the $\hat{p}_z$ component is zero.

Write a code to obtain the Landau levels $E_n(B)$ of the particle in a homogeneous magnetic field $\boldsymbol{B} = (0, 0, B)$ for a given value of S. What values can the index n take? Find the analytical expressions for the Landau levels in case of $S = 1/2,\ 1,\ 3/2,\ 2$.

Hint: see problem 33 of the Rudolf Ortvay problem-solving competition in 2011.

(József Cserti)

30. A quantum particle with a quasi-spin degree of freedom can be described by a wave function of N components. Thus the Hamiltonian $\hat{\mathbf{H}}$ is a matrix of size $N \times N$. Its components are denoted by $\hat{H}_{kl}$. The Hamiltonian is hermitian, i.e. $\hat{H}^{\dagger}_{kl} = \hat{H}_{lk}$.

Due to the coupling between the quasi-spin and the orbital motion of the particle the components of the Hamiltonian are functions of the position operator $\hat{\boldsymbol{x}}$ and momentum operator $\hat{\boldsymbol{p}}$.

More precisely—as the careful analysis of the experimental data has shown—the components of the Hamiltonian do not depend separately on the position and momentum operators but they are functions of a specific combination of them: functions of the vector operator $\hat{\boldsymbol{c}} = \hat{\boldsymbol{x}} + \alpha\, \hat{\boldsymbol{p}}$, i.e. $\hat{H}_{kl}(\hat{\boldsymbol{c}})$. Here α is a real constant, the value of which can be determined experimentally.

Problem: Study the system in Heisenberg picture. Give the momentum operator $\hat{\boldsymbol{p}}(t)$ as the function of the time. Explain the results.

Hints:

—At first step introduce the auxiliary vector operator $\hat{\boldsymbol{b}} = \beta\, \hat{\boldsymbol{x}} + \gamma\, \hat{\boldsymbol{p}}$ which is linearly independent from vector $\hat{\boldsymbol{c}}$, where the real coefficients β and γ can be chosen in such a way that the commutation relation between the operators $\hat{\boldsymbol{c}}$ and $\hat{\boldsymbol{b}}$ has the simplest form.

—Then calculate the commutator of the operator $\hat{\boldsymbol{b}}$ with an arbitrary analytical function $f(\hat{\boldsymbol{c}})$ of the operator $\hat{\boldsymbol{c}}$.

—The time developing operator $\hat{\mathbf{G}}(t)$ can be expressed using the projector deconvolution of the Hamiltonian $\hat{\mathbf{H}}$.

(Gyula Dávid)

31. Samuel L. Braunstein and H. J. Kimble, in their article on quantum teleportation published under the title 'Teleportation of Continuous Quantum Variables', refer to John Stewart Bell's fundamental work in quantum information theory and claim that the following function in phase space named after Einstein, Rosen, and Podolsky

$$W_{\mathrm{EPR}}(\alpha_1; \alpha_2) = \frac{4}{\pi^2} \exp\left\{-\mathrm{e}^{-2r}\left[(x_1 - x_2)^2 + (p_1 + p_2)^2\right] - \mathrm{e}^{2r}\left[(x_1 + x_2)^2 + (p_1 - p_2)^2\right]\right\},$$

is a Wigner function, where $\alpha_j = x_j + \mathrm{i}p_j$, i is the imaginary unit, and $r \geq 0$.

Show that, contrary to the authors' claim, the function shown here is not a Wigner function.

(Gábor Homa)

32. In a distant corner of the universe, on the planet of elastic interactions, scientists have long known that the time-dependent psycho-social relationships of citizens can be well described by a Heisenberg equation of motion. The Hamiltonian that determines the dynamics between two specific individuals A and B is given by

$$\hat{H} = \hat{\boldsymbol{r}}^{\mathsf{T}} \cdot \mathbf{M} \cdot \hat{\boldsymbol{r}} + \boldsymbol{K}^{\mathsf{T}} \cdot \hat{\boldsymbol{r}} + C,$$

where

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 86 & 2 \\ -1 & 3 & -48 & -46 \\ 86 & -48 & 4588 & 496 \\ 2 & -46 & 496 & 1772 \end{pmatrix}, \qquad \boldsymbol{K} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \qquad C = 5,$$

and the components of the operator $\hat{\boldsymbol{r}}$ are coordinate and momentum-like operators with the following properties:

$$\hat{\boldsymbol{r}} = \begin{pmatrix} \hat{x}_A \\ \hat{x}_B \\ \hat{p}_A \\ \hat{p}_B \end{pmatrix}, \quad [\hat{x}_m, \hat{x}_n] = 0, \quad [\hat{x}_m, \hat{p}_n] = i\delta_{mn}, \quad [\hat{p}_m, \hat{p}_n] = 0, \quad m, n = A, B.$$

a) In practice local scientists use the following simpler form of the operator $\hat{H}$:

$$\hat{H} = (\hat{\boldsymbol{r}} - \boldsymbol{r}_0)^{\mathsf{T}} \cdot \mathbf{M} \cdot (\hat{\boldsymbol{r}} - \boldsymbol{r}_0) + D.$$

What are the values of the vector $\boldsymbol{r}_0$ and the scalar D?

b) As the population of the planet increases, it becomes more and more expensive for the planet's statistical office to track binary relationships due to the deteriorating economic situation. They decide to switch to new, cheaper $\hat{\boldsymbol{s}}$ indicators by a transformation $\hat{\boldsymbol{s}} = \boldsymbol{f}(\hat{\boldsymbol{r}})$. However, the planetary governor decrees, for the public good, that

- Only canonical transformations $\boldsymbol{f}$ are allowed.
- In order to reduce the state costs, the Hamiltonian expressed by the new transformed indicators must have the form
  $$\hat{H} = \hat{\boldsymbol{s}}^{\mathsf{T}} \cdot \mathbf{N} \cdot \hat{\boldsymbol{s}} + D,$$
  where the matrix $\mathbf{N}$ must be diagonal.
- The old and new indicators must be explicitly convertible from each other, so the mapping $\boldsymbol{f}$ must be invertible.

Give a concrete realization for the mapping $\boldsymbol{f}$. If there are several possible transformations, how can they be derived from this specific mapping? What are the diagonal elements of $\mathbf{N}$?

(András Csordás and Gábor Homa)

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