THE 46th—18th INTERNATIONAL—RUDOLF ORTVAY PROBLEM SOLVING CONTEST IN PHYSICS
22 October—2 November 2015

1. Nowadays, when you have to pay for everything, you still have the opportunity to stand into the shadow of a house at a hot summer day for free. Imagine that as a new landlord we would like to change that and prevent anyone to enjoy the shadow of our prospective building. So we would like to construct a building that has no shadow. Is it possible? If so, what is the shape of such a building with the maximal volume for a given construction site area? Suppose that when the Sun lies lower than 5 degrees it is not too hot, so we are not interested whether our construction has a shadow or not. Investigate the problem for differently shaped construction sites, different geographical latitudes and for different periods of the day and the year.

(Gergely Dálya and Bence Bécsy)

2. One fifth of the surface of an exactly spherical fictitious planet is not covered by the geostationary satellite's TV broadcast. Denote the maximal deviation between the the vertical direction (defined by a hanging plummet) and the radial direction by δ. Calculate the sine of δ.

(Gyula Dávid)

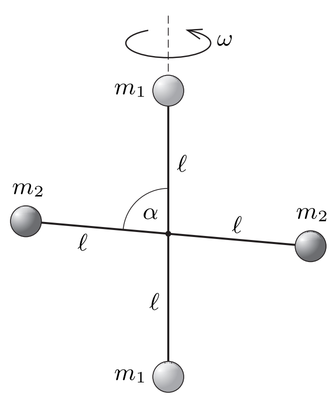
3. A small body is moving on a slope of inclination angle α, its initial velocity $\mathbf{v}_0$ is perpendicular to the gradient of the slope, and the coefficient of the friction is just $\mu = \tan\alpha$. Describe the motion, and give its long time asymptotics, i.e., the form and position of the trajectory and the final velocity. (Hint: find an appropriate parametrization for the different quantities.)

(Ferenc Woynarovich)

4. A solid sphere of radius R is fixed on a horizontal floor. We pour fine sand onto the sphere (not necessarily to the topmost point) in a thin, vertical stream. The sand particles slip on the surface without friction, at some point they leave the surface and drop on the horizontal floor. Let us model the impact of the sand particles on the sphere in the following way: the particle, having velocity v, will retain the magnitude of its velocity but continues in a randomized direction in the tangential plane from the impact point. Determine the curve drawn by the boundary of the sand-covered region. Study this curve as a function of the impact point of the vertical sand stream and the height of the pouring.

(Gyula Dávid and József Cserti)

5. Pairs of equal pointlike masses are fixed on ends of two massless rods of length 2ℓ, such that it results in two 'dumbbells', weighting $2m_1$ and $2m_2$, respectively. The rods are attached together rigidly in their midpoints such that the angle between the rods is α. The system is then rotated around the axis going through the two masses m_1 with angular velocity ω (in the state of weightlessness) and released. Describe the motion, as a function of the angle and ratio of the masses, in the approximation that α is close to $\pi/2$. Study the cases when $m_1 < m_2$ and $m_1 > m_2$.



(Gergely Fejős and Máté Vigh)

6. In the framework of classical mechanics, we require the *first variation* of the action to vanish when deriving the equations of motion. This merely means that the solution, i.e., time dependent coordinate function is a stationary point of the action. This information however does not imply that the action has an extremum, and whether this is actually a minimum or maximum. To decide on that, the *second variation* is to be calculated. Let us study this problem.

   a) Consider systems with one degree of freedom. The Lagrangian takes the form of $L(t, q(t), \dot{q}(t))$ where the *dot* represents the time derivative and the action is given by the following integral:

$$S = \int_{t_1}^{t_2} \mathrm{d}t \; L(t, q(t), \dot{q}(t)).$$

   Show that the second variation of S at fixed boundaries t_1 and t_2 and with variations $\delta q(t)$ which vanish at the boundaries, can be written in the form

$$\delta^{(2)} S = \int_{t_1}^{t_2} \mathrm{d}t \left[A(\delta q)^2 + B(\delta \dot{q})^2 \right],$$

   where A and B depends on time, on the coordinates q and on the time derivatives of q. Determine the coefficients A and B.

   b) Now study the one dimensional motion of a pointlike particle in external potential, that is, with Lagrangian of the form $L = \frac{m\dot{x}^2}{2} - V(x)$. Show that the second variation of the action can be written as:

$$\delta^{(2)} S = \int \mathrm{d}t \; \delta x \hat{M} \delta x.$$

   Determine the operator $\hat{M}$. Which is the condition that $\hat{M}$ needs to fulfill in order the action to be minimal or maximal?

   c) Calculate the second variation of the action of the harmonic oscillator $\left(L = \frac{m\dot{x}^2}{2} - \frac{m\omega_0^2 x^2}{2}\right)$. Is it true that the action is minimal? If not, find such a variation $\delta x(t)$, which reduces the value of the action (within some integration limits).

   d) What can one conclude about the extremal value of the action in the case of a pointlike particle moving in one dimension in an external potential?

(Áron Kovács)

7. Find the general solution of the following wave equation:

$$\frac{\partial^2 \Phi}{\partial t^2} + 2\alpha \frac{\partial \Phi}{\partial t} + \alpha^2 \Phi = c^2 \frac{\partial^2 \Phi}{\partial x^2}.$$

   How can you characterize the solutions?

(Géza Tichy)

8. A thermodynamic engine undergoes a quasi-static repetitive cycle which has a diagram in a shape of an ellipse with axes parallel to the coordinate axes. The elliptic diagram appears on the

   a) temperature – entropy plane (for any material)

   b) enthalpy – entropy plane (for ideal gas)

   c) pressure – volume plane (for ideal gas).

   For each cases, give an upper limit for the thermal efficiency of the cycles.

(Gyula Radnai)

9. Assume that the thermal energy and the equation of state of a spring are given by $E = CT + f(x)$ and $F = g(x) - BT$ where E is the energy of the spring, F is the force acting on the spring, T is the temperature of the spring, x is the displacement of the spring from its equilibrium position, B and C are constants, finally $f(x)$ and $g(x)$ are two given functions of x.

a) What the relation between the two functions $f(x)$ and $g(x)$ should be to satisfy the well-known result for the efficiency of the Carnot-cycle, namely

$$\eta = \frac{T_2 - T_1}{T_2},$$

where T_1 and T_2 are the absolute temperature of the cold and hot reservoir, respectively?

b) Consider two situations. In the first case the spring is standing on the ground and a body with mass m is placed on the spring. In the second case a body with mass m is suspended by the hanging spring. What is the heat capacity in both cases?

c) What is the result in question b) if $f(x) = \frac{1}{2} Dx^2$ where D is a constant?

(Géza Tichy)

10. 'It is impossible, by means of inanimate material agency, to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects.' (Kelvin) 'Heat can never pass from a colder to a warmer body without some other change, connected therewith, occurring at the same time.' (Clausius)

By sticking to the letter of this law, show that it is impossible to arrange lenses, mirrors, optical fibers, or any other similar devices in such a way that they can focus sunlight on a body so that it becomes hotter than the surface of the Sun.

By sticking to the letter of the law, we do not mean that an in-depth analysis of the wording of Kelvin or Clausius is necessary for the solution we have in mind. What we mean is that we should refrain from using concepts, such as the entropy, that don't occur in their formulation of the law. Some say that the content of the second law of thermodynamics is the existence of the entropy, that is, an additive and extensive function of state which never decreases in an adiabatic process, and it increases precisely when the process is irreversible. But our problem is to decide what is possible adiabatically in a particular system, so an argument that relies on the notion of entropy risks the fallacy of begging the question.

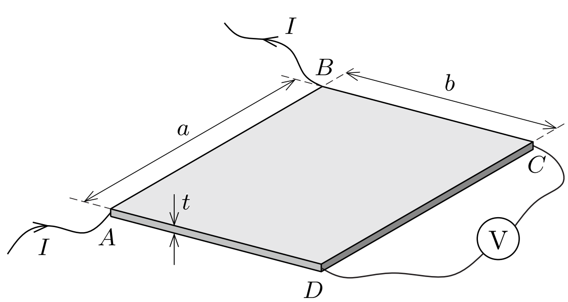
Also, note that when we apply Kelvin or Clausius' law, we have to make sure that at the beginning and in the end of the process the system consists of components each of which is in an equilibrium thermodynamic state. During the state evolution the components can engage in all kinds of interactions, for example, they can exchange radiation. But if there remains some uncaptured radiation in the final state, then it is a component which cannot be characterized by a thermodynamic state, and therefore the entire system is not (directly) within the scope of validity of the law. But if all radiation is absorbed by some thermodynamic component or captured and turned into say a box of photon gas, it will not pose any obstacle to the application of the law. The Sun is radiating, so in order to prove our claim from the second law, the actual system needs to be related to a modified system, to which the law can be applied without any difficulties, and from whose properties we can infer something about the original system.

(Szilárd Farkas and Zoltán Zimborás)

11. A metallic torus with minor radius r and major radius R is on ground potential. Determine the force exerted on a pointlike charge q which is located on the rotational axis of the torus, at a distance of h from the midpoint.

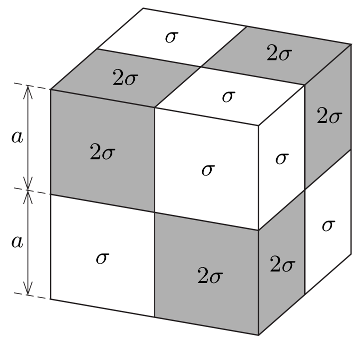
(Gábor Széchenyi)

12. Given a rectangular-shaped metallic plate of side-lengths a and b, thickness of t ($t \ll a, b$), conductivity of σ. If current I enters the plate at the corner A, and leaves it at corner B, what is the reading on the voltmeter connected to the corners C and D? Find the numerical values of the voltage for $b = a$ and $b = a/2$, respectively.



(Máté Vigh)

13. A $2 \times 2 \times 2$ Rubik's cube consists of eight solid, uniform, metallic cubes of side-length a. Four cubes out of the eight have conductivity 2σ, while the conductivity of the other four cubes is σ. The cubes with different conductivities have no common faces (see the figure). Two metallic plates of very good conductivity are attached to the two opposite sides of the Rubik's cube (not shown in the figure). Find the total current flowing between the two plates if a battery of voltage U_0 is connected to them.



(Péter Gnädig and Máté Vigh)

14. Two arbitrary points of a very thin spherical shell is connected by a thin straight conductor. A constant current I flows in this conducting wire, with the circuit being closed by the charge returning on the surface of the sphere. Determine the magnetic field created by the current flow, both inside and outside of the shell.

(László Sasvári and Máté Vigh)

15. Let us consider an infinite plain with a rotationally symmetric charge distribution of the form

$$\varrho(\mathbf{r}) = \frac{Qd}{(r^2 + d^2)^{3/2}} \delta(z),$$

where r is the distance of the charge on the plane from the origin, z is the distance perpendicular to the plane and d is a constant with dimension of length. Assuming that the charge distribution rotates with angular velocity ω around the axis z, calculate the magnetic induction vector field $\mathbf{B}(\mathbf{r})$. Give the solution in a closed form. Discuss the special properties of the field configuration.

(László Oroszlány)

16. In the fictitious realm of 'Anisotropia', the 'keep right' rule is taken so seriously that, in respect of the 'International Year of Light', the police has issued a law, requiring the light to move twice faster towards the right, than towards the left. It is indeed fortunate that in Anisotropia, there is only one space dimension, so the other directions are irrelevant. Once the rule has been announced, all rulers, all watches and all pairs of twins are required to behave in accordance. In fact, the police is in a deep respect of Einstein, so the principles of relativity are still valid, and the relevant rules hold in all inertial systems. The speed of light, though different in the two directions, is constant with respect to all inertial systems.

Let us investigate the rules of special relativity theory in Anisotropia. Find the matrices corresponding to the Lorentz transformation. Let us show that the set of these anisotropic Lorentz transformations forms a group as well. Find the canonical parametrization, and express the elements of the matrices with these parameters. Determine the formulae of summing and subtracting velocities. Which is the maximum relative velocity between any two inertial systems?

Let us write down the 1+1 dimensional (scalar) wave equation, which respects the unusual laws in Anisotropia.

(Enthusiasts may even work out the more general theory, in which the velocity of light moving towards the right is c_+, and moving towards the left is c_- where the two values are arbitrary but not equal.)

The well known scientific skeptic, Mynden Lee Ben Canal, is shaking his head, stating that the physics discussed above is not new at all, but isomorphic with a commonly known theory. Is that statement right?

(Gyula Dávid)

17. Solve the eigenvalue problem of the electromagnetic field tensor $F_{kl} = \partial_k A_l - \partial_l A_k$, where $A_k(x)$ is the four-vector of the electromagnetic potentials. What is the physical meaning of the eigenvalues and the eigenvectors? Discuss the special singular cases.

(Gyula Dávid)

18. Before the formulation of the general theory of relativity, description of the effects of gravity was attempted by introducing a scalar gravitational potential $\Phi(x)$ within the framework of the special relativity theory. The 'gravitational four-force' appears, in the classical analogy, as the 'mass × gradient of gravitational potential'. This means that the motion equation in this theory is as follows:

$$\frac{d}{d\tau}(Mu_k) = M\,\partial_k\Phi(x),$$

where M is the rest mass of the particle, u_k is the four-vector of velocity (normalized to c), τ is the proper time.

Let us study the problem of vertical free fall in homogeneous gravitational field in this theory, that is, let the gravitational potential Φ be $\Phi = gz$, where z is the vertical coordinate, g is the usual gravitational acceleration. For simplicity, let us work in the unit system of $c = g = 1$. Assume that the particle is released at $t = 0$ from height of H, starting with zero initial velocity.

Let us start with measuring the proper time τ at this instant. Calculate the rapidity ω as a function of τ, and based on this determine the three-velocity v and coordinate z as functions of the system time t and proper time τ. At most for how long time can the particle enjoy the experience of free fall? Is it possible that within finite time (system or proper) the particle reaches the velocity of light? If the falling body starts from height H, at which speed will it impact the ground, and which is the three-momentum that it transfers?

Let us study the non-relativistic limit of the results (for this, bring back the dimensional c and g quantities), and let us show that we arrive back to the Newtonian laws of physics. If g is the gravitational acceleration on Earth, from how high one should release the particle, so that the difference between the relativistic and classical results becomes one per mil?

(Gyula Dávid)

19. The equivalence principle is one of the fundamental concepts of the general theory of relativity by Albert Einstein: it states that gravity and acceleration can not be differentiated by any local measurements. Wilbert Zweistein, on the other hand, states that with a thermometer, as well as a probe mass attached to a spring-operated force gauge, he can locally decide if he is observing the gravitational attraction of a large mass (e.g. a star) or of the acceleration of his own lab. What can he mean with the statement, and is he possibly right?

(Gergely Fejős)

20. Let us prove, that if in the framework of general theory of relativity the space-time is 'static', or in other words, the components of the metric tensor do not depend on the zeroth (time) coordinate, as well as if the space/time mixing components $g_{0\alpha}$ are all zeros, then the geodetic equations of motion of a free falling body under gravitational influence can be obtained from the Lagrangian of classical mechanics with the form $L = K - V$. The general coordinates in L are the components x^α ($\alpha = 1, 2, 3$) of the position vector $\mathbf{r}$, which are considered as functions of the system time t (not that of the proper time τ).

The kinetic energy K is a quadratic function of the velocity components $\dot{x}^\alpha$, whereas the quantity $V(\mathbf{r})$ is some function of the position vector $\mathbf{r}$. Determine the Hesse matrix, containing the coefficients of the kinetic energy, and also the 'effective gravitational potential energy' $V(\mathbf{r})$. What is the relation of this Lagrangian defined above with the usual covariant Lagrangian in the general theory of relativity, namely with $(-mc^2)$?

How does the situation change if the particle is not free, but moves in a (given) external field, e.g., in electromagnetic or scalar field?

(Gyula Dávid)

21. Bungee jumping into a black hole. Alice and Bob organize an expedition to explore the interior of a supermassive black hole and hope to return back out as follows. In separate spaceships they slowly descend radially towards the horizon of Schwarzschild black hole. At radius $1.01\,r_S$ where $r_S = 2GM/c^2$ is the Schwarzschild radius, they synchronize their clocks to $\tau_A = \tau_B = 0$, Bob turns off the engines, and starts to fall radially inwards. He crosses r_S at proper time τ_{B1}. Alice remains at $1.01\,r_S$, waits the same amount of proper time $\tau_{A1} = \tau_{B1}$, and then releases a thin spherical shell of electrons of charge Q concentric with the black hole radially inwards with initial velocity v with the intention of saving Bob.

The Reissner–Nordström line element of a charged static and isotropic black hole of mass M and charge Q is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

in units $G = c = 4\pi\epsilon_0 = 1$.

a) What is the radial coordinate of the event horizon outside of the spherical shell after the spherical shell reaches the singularity?

b) How large should v be for the shell to catch up with Bob before he reaches the singularity?

c) Is it possible for Bob to return to Alice for a suitable range of Q and v?

Let us neglect tidal effects, electromagnetic and gravitational radiation, quantum effects, the interstellar medium, and human limitations with respect to acceleration.

(Bence Kocsis)

22. Relativistic jets are among the most spectacular phenomena in high energy astrophysics. Observations show highly energetic particles emerging from the vicinity of black holes in a conical geometry where the particles' Lorentz factor may reach $\Gamma = 1000$. The origin of such high velocity particles is not well understood, they are believed to be accelerated by the interaction of a spinning black hole and the magnetic field generated by a gaseous disk. Let us examine instead if the central object may be a maximally extended Kerr space-time. The maximally extended Kerr space-time describes the unique stationary solution to the vacuum Einstein's equations around a spinning object of mass M. It is 'extended' because the black hole provides a gateway to other asymptotically flat universes. Observers in one universe may enter the black hole and exit into another universe on time-like world lines without touching the ring-shaped singularity. What conditions must the conserved quantities (energy E, the z-component of the angular momentum L_z, and the Carter constant K) of photons and massive particles satisfy for them to enter the horizon, avoid hitting the singularity, and escape to infinity in another universe? What is the image of such outgoing particles for static observers at infinity of this other universe as a function of energy and the Kerr black hole spin parameter?

    (Let us ignore quantum and electrodynamical effects and the backreaction of the particles gravity on the space-time, and assume that massive particles and photons are point-like.)

    (Bence Kocsis)

23. An ensemble of identical atoms is given. Show, the binding energy per atom that is the cohesive energy is approximately proportional to the square root of the coordination number.

    (Géza Tichy)

24. Let A and B be two arbitrary, non-commuting 2×2 Hermitian matrices. Let us construct all normalized state vector(s), for which the Heisenberg's uncertainty relation corresponding to the operators A and B is fulfilled as an equality.

    (Gyula Dávid)

25. Consider the simplest quantum model for a particle propagating along a wire and scattering off a pointlike potential. The corresponding time-dependent Schrödinger equation reads as

$$i\hbar\, \partial_t\, \Psi(x,t) \;=\; -\frac{\hbar^2}{2m}\,\partial_x^2\,\Psi(x,t) + g\,\delta(x)\,\Psi(x,t),$$

    where g is the coupling strength of the Dirac-delta potential. If in appropriate units $g \in \mathbb{R}$, and $g > 0$, we have a repulsive scattering potential, whereas for $g < 0$ the potential is attractive.

    First let us construct the eigenstates with positive energy, i.e., the scattering states, then calculate the amplitudes of reflection and transmission as functions of the energy, g, and m – this is an elementary exercise. Next show that the scattering states form a complete basis. What is the situation for $g < 0$?

    If $g \in \mathbb{C}$, the time-dependent Schrödinger equation remains meaningful and well defined. For $\operatorname{Im} g < 0$, the norm of the wavefunction decreases over time. This describes a situation where the potential can also scatter the particle out of the wire, while the norm of the wavefunction corresponds to the probability that the particle remains within the wire. What are the probabilities of transmission and reflection now? Plot the result with typical values as function of $\operatorname{Im} g$. For what value of $\operatorname{Im} g$ is it most likely that the particle is scattered out of the wire? What can we say about the set of scattering states in the case of complex g, with $\operatorname{Im} g < 0$? Can you answer the same questions for $\operatorname{Im} g > 0$?

    (János Asbóth and Géza Györgyi)

26. Researchers observed that molecular hydrogen (H_2) forms from atomic hydrogen (H) in deep space provided the process is facilitated by the intermediaries (gas molecules, ice crystals and contaminants) present in this rarefied medium. These catalysts are capable of adsorbing the H atoms, which subsequently migrate from one lattice site to another via tunneling. When two H atoms collide on one site, the reaction $H + H \to H_2 + \gamma$ may take place, and finally the (H_2) molecule exits the crystal, whereas the latter absorbs the energy of the photon produced in the reaction ($E_\gamma \approx 4.5$ eV).

    To follow this process, assume a greatly simplified model where at $t = 0$ a H atom is found at site s_0 of a (square) lattice (see figure), from where it cannot move on in time anywhere but to one of the four adjacent lattice sites denoted by s_1, s_2, s_3 and s_4.

    If we start by ignoring the possibility of tunneling to one of the lattice sites shown above, then the states denoted by $|\varphi_i\rangle$ (where the atom is found near the given site and $i = 0, \ldots, 4$) are the orthonormal eigenstates of the unperturbed Hamiltonian $\hat{H}_0$ belonging to the same energy, say E_0.

The coupling between the state $|\varphi_0\rangle$ and the states $|\varphi_k\rangle$ modifies the Hamiltonian; so we need to add to $\hat{H}_0$ a perturbing term, $\hat{H}_1$ defined by

$$\hat{H}_1\,|\varphi_0\rangle = -a\,(|\varphi_1\rangle\ + |\varphi_2\rangle\ + |\varphi_3\rangle\ + |\varphi_4\rangle\ ),$$

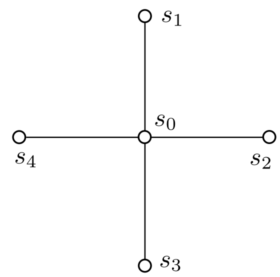
$$\hat{H}_1\,|\varphi_k\rangle = -a\,|\varphi_0\rangle$$

where a is some real, positive constant and $k\ =\ 1,\ 2,\ 3,\ 4$. All other possible couplings are neglected. Solve this problem to derive the quantum dynamics of the model.

a) Find a suitable representation for the orthonormal eigenfunctions of the full Hamiltonian $\hat{H}$, and calculate the corresponding energies, wave functions and degeneracies.

b) Assume, as before, that at $t = 0$ the H atom is located at site s_0. Write down the wave function of the atom, $|\varphi(t)\rangle$, at some arbitrary later time instant t. What is the time period T required after which we can 'safely' declare that the atom has changed its lattice position?

c) Determine the numerical value of T by taking a suitable value for the parameter a above.



(Péter Magyar)

27. Let us have N entangled spins in the state of $\frac{1}{\sqrt{2}}\left(|\uparrow_1\uparrow_2\dots\uparrow_N\rangle + |\downarrow_1\downarrow_2\dots\downarrow_N\rangle\right)$ which are sent out to N number of observers. The i-th observer will receive the spin state with a Stern-Gerlach analyzer set in the direction of $\mathbf{t}_i$, and measures the spin projection with the result of ± 1 in units of $\hbar/2$. Let P_{ij} denote the probability that the i-th and the j-th observer measured the same value. What is the minimum of the quantity given by $\sum_{i=1}^{N}\sum_{j=1}^{N} P_{ij}$?

How should the observers set their analyzers to reach this minimum? For simplicity, assume that the vectors $\mathbf{t}_i$ for all observers lie within the $x-z$ plane.

What happens if instead of the entangled ones, we send out spins polarized in the $+z$ direction $(|\uparrow_1\uparrow_2\dots\uparrow_N\rangle)$ with probability $1/2$, and spins polarized in the $-z$ direction $(|\downarrow_1\downarrow_2\dots\downarrow_N\rangle)$ with probability $1/2$? What is the minimum of the quantity defined above? In this case how to set the polarizers in the $x-z$ plane?

(Gábor Széchenyi)

28. Sometimes path integrals can facilitate a nonrigorous demonstration of some mathematical relation whose precise derivation can be quite laborious. Here we study a problem in stochastic calculus which can be solved with relative ease by the aid of functional integration.

Consider a particle whose position R on the real line satisfies the following first order stochastic differential equation:

$$\dot{R} = F(R) + \xi. \tag{1}$$

Here F is a smooth function, $\dot{R}$ is the time derivative of R, and ξ is a white noise. More specifically, ξ is a Gaussian noise, so if $G[\xi]$ is some functional of ξ, then its expectation value is

$$\langle G\rangle = \int \mathcal{D}\xi\ \exp\left(-\tfrac{1}{\eta}\textstyle\int \mathrm{d}t\,\xi(t)^2\right)\ G[\xi], \tag{2}$$

where $\mathcal{D}\xi$ denotes integration with respect to the usual (appropriately normalized) path integral measure. In the Ito interpretation, the discretized version of equation (1) takes the form

$$\frac{R_{i+1} - R_i}{\Delta t} = F(R_i) + \xi_i, \tag{3}$$

where the subscripts are the discrete time indices, and Δt is the length of one time step. If $G[\xi]$ depends only on the values of ξ in a bounded time interval, then its expectation value in the discrete setting can be expressed in terms of an ordinary finite dimensional integral. In fact, one way to make sense of the path integral in equation (2) is by taking the continuum limit of such discretized expectation values.

Let $p(t,x)$ be the probability density of the particle's position at time t. Starting with the discrete setting and then taking the continuum limit $\Delta t \to 0$, prove that p satisfies the following parabolic partial differential equation:

$$\frac{\partial p}{\partial t} = \frac{\eta}{4}\frac{\partial^2 p}{\partial x^2} - \frac{\partial}{\partial x}(Fp). \tag{4}$$

This is the Fokker–Planck equation. In the Stratonovich interpretation, the discretized equation of motion is

$$\frac{R_{i+1} - R_i}{\Delta t} = F\left(\frac{R_{i+1} + R_i}{2}\right) + \xi_i. \tag{5}$$

By going through the derivation for the Ito case, demonstrate that despite the modifications that the Stratonovich discretization may necessitate, eventually we get the same Fokker-Planck equation as in the Ito case.

(Szilárd Farkas and Zoltán Zimborás)

29. Consider the $O(N)$-invariant N-component scalar field theory in 3+1 dimensions, in a version which is renormalizable according to dimension power counting, with the Lagrangian

$$L = -\frac{1}{2}\,\Phi_i(\Box + m^2)\Phi_i - \frac{\lambda}{24N}\,(\Phi_i\Phi_i)^2,$$

where $m > 0$ and $\lambda > 0$ are positive constants.

Now consider the perturbative series of self-energy at any value of λ in case of $N \gg 1$. Display all the graphs which give a contribution in the leading, as well as in the next-to-leading order for the terms of the series in powers of $1/N$.

After Wick rotation and in the cutoff regularization scheme determine the counter-terms of the mass as well as the coupling constant in leading and in the next-to-leading order.

Plot the renormalized self-energy at a given momentum value as a function of the cutoff at various values of λ. What can one conclude about the renormalizability of the model?

(Gergely Fejős)

30. Consider the following simple fund-management model:

Every day, the market goes up ($x_t = +1$) with a probability of p_t, or it goes down ($x_t = -1$) with a probability of $(1 - p_t)$. Each morning, the fund manager can decide what r_t ratio of the fund's wealth he wants to invest.

The evolution of the W_t wealth is described by the following map ($W_1 = 1$):

$$W_{t+1} \;=\; W_t\,(1 + r_t\,x_t).$$

The investment rate r_t can be negative: This corresponds to 'shorting' the market, in this case we bet on the market going down.

We can assume the following about the the stochastic probability-series p_t: Initially, it is between 0.5 and 0.7. It is quasi-continuous, that is, its daily change is never more than 0.01. Its value always remains between 0.4 and 0.8.

Our task as fund-manager is to determine the investment rate r_t each morning. We will use the following adaptive strategy:

$$r_1 = r,$$

$$r_{t+1} = max\,(-0.5;\, min\,(0.9;\, a\,r_t + b\,x_t + c)).$$

The min and max functions establish regulatory constraints: We can never invest more than 90 % of the fund's wealth, and we cannot short more than 50 % of the wealth. The four parameters (r, a, b, c) specify our strategy. Our goal is to outperform the competing funds on a one-year horizon (this means $T = 250$ trading days), that is, we want to achieve a high W_{250} result compared to our competitors.

We compete with ten other funds. Nine of these are passively managed; they work with the following constant rates: $r = 0$; 0.1; 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8 (For these, the other adaptive strategy parameters are: $a = 1$; $b = 0$; $c = 0$.) The tenth competitor is actively managed, it uses an adaptive strategy as defined above. Its fund manager is the problem poser.

At the end of the year, investors rank the funds by results. They give the best fund 100 points, the second 90, the third 80, and so on. The worst fund gets 0 points. Our goal is to maximize our expected points (that is, to minimize our expected rank). How should we specify our adaptive strategy?

50 % of the mark given for the solution will be based on the actual result of the specified strategy. I will estimate the expected popularity point by averaging 100,000 test runs.

I ask the contestants to clearly write down their (r, a, b, c) specification at the top of their solution sheets. The other 50 % will be given for the presentation of the solution.

Key words: '*kelly criterion*', '*kalman filter*'.

(Zsolt Bihary)

31. Most infrared (IR) cameras operate in the wavelength range of 7.5–14 microns, which is the so called 'atmospheric IR window'. In this range the atmosphere is mostly transparent. However, a camera in a vertical position observes around (-60) – (-40) Centigrade temperatures for the clear sky (no clouds at all) in spite of a few Kelvins (temperature of the empty space). This is due to the presence of water vapor which (weakly) radiates in this wavelength range, and the intensity is determined by the 'integrated water column' (total amount of water expressed in units of mm). At increasing zenith angles, the observed temperature is gradually increases (virtual warming).

    The task is to determine the pattern of virtual temperature of the clear sky as a function of the zenith and the azimuth angles.

    Basic assumptions: a) 100 % of the water vapor is located in the troposphere, b) The concentration of water vapor is homogeneous in the troposphere (this is far from the reality, but we neglect here the strong variability), c) The water vapor does not only radiate but also absorbs IR radiation. We assume that the optical depth is constant (say 5000 m).

    Supplementary question: how large is the error when we neglect the curvature of Earth?

(Imre Jánosi)

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