THE 36th—8th INTERNATIONAL—
RUDOLF ORTVAY
PROBLEM SOLVING CONTEST IN PHYSICS
2005—100th Anniversary of Einstein's Miraculous Year

The Physics Students' Association of Eötvös University, Budapest, the Roland Eötvös Physical Society and the Hungarian Association of Physics Students proudly announce the **38th** – and for the eighth time **international** Rudolf Ortvay Problem Solving Contest in Physics, between 28 October 2005 and 7 November 2005.

2005 is the World Year of Physics, commemorating the epoch-making discoveries of Albert Einstein 100 years ago. That is why this year's Ortvay contest is also dedicated to the memory of Einstein: each problem is related to his oeuvre, to the questions posed and/or solved by him, or to new developments in these fields.

Every university student from any country can participate in the Ortvay Contest. PhD students compete in a separate category. The contest is for individuals: solutions sent by groups of students are not accepted. The name, the university, the major, and the university year should be indicated on the solutions. Pseudonyms and passwords cannot be used: each contestant has to use his/her own name.

The problems can be **downloaded** from the webpages of the Ortvay Contest

http://ortvay.elte.hu/

in Hungarian and English languages, in html, LaTeX and Postscript formats, from **12 o'clock (Central European Time, 11:00 GMT), Friday, 28 October 2005**. The problems will also be distributed by local organizers at many universities outside of Hungary.

Despite all the efforts of the organizers, it may happen that some unclear points or misprintz stay in the text. Therefore it is very useful to visit the webpage of the contest from time to time, as the corrections and/or modifications will appear there.

Each contestant can send solutions for up to 10 problems. For the solution of each problem 100 points can be given.

Any kind of reference material may be consulted; textbooks and articles of journals can be cited.

Each problem should be presented on (a) separate A4, or letter-sized sheet(s). The contestants are kindly asked to use only one side of each sheet. Solutions written by pencil or written on thin copy-paper will not be accepted as these cannot be faxed to the referees.

Computer programs appended to the solutions should be accompanied by detailed descriptions (what computer language it has been written in, how to use the program, which parameters can be set, what notations are used, how to interpret the output figures and graphs, etc.) They can be enclosed on floppy disks, or sent via email to the addresses below.

Solutions can be sent by mail, fax, or email (in LaTeX, TeX, pdf or Postscript formats. Contestants are asked not to use very special LaTeX style files unless included in the sent file(s). Electronically submitted solutions should be accompanied—in a separate e-mail—by the contents and, if necessary, a description explaining how to open it.

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Deadline for sending the solutions: 12 o'clock CET (11:00 GMT), 7 November 2005.

Contestants are asked to fill in the form available on our webpage after posting their solutions. It will be used for identification of contestants and their solutions. **Without filling in the form, the organizers cannot accept the solutions! The form is available only on 7 to 8 November.**

The contest will be evaluated separately for each university year, according to the total number of points. The referees reserve the right to withhold, to multiply or to share some prizes. Beyond the money prizes given for the first, second, and third places, honorable mentions and special prizes for the outstanding solutions of individual problems can be awarded. This is why it is worthwhile sending even one or two solutions.

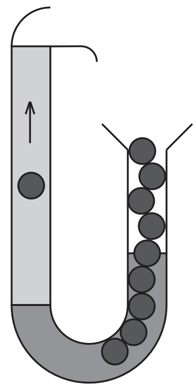
The announcement of the results will take place on 8 December, 2005. The detailed results will be available on the webpage of the contest thereafter. Certificates and prizes will be sent by mail.

We plan to publish the assigned problems and their solutions in English language—to which the contribution of the most successful participants is kindly asked. The volume is planned to be distributed all over the world with the help of the International Association of Physics Students, as well as the contestants themselves. We hope this will help in making the contest even more international.

Wishing a successful contest to all our participants,

the Organizing Committee:
Gyula Dávid, Attila Piróth, József Cserti

1. It is well known that during his miraculous year Einstein was working as an examiner in the Swiss Patent Office in Bern. One day an inventor knocked on his door with his perpetual motion machine. The heart of the assembly was a U-tube, with its left arm longer than the right one. The left arm was filled with water, while the right one—with a denser liquid.

   Just like in other decent communicating vessels, the liquid level in the left arm was higher than in the right one. The assembly also contained some wooden balls whose density was smaller than that of either liquid. There were several balls in the right arm, and since their diameter was only slightly smaller than that of the tube, they were stacked on top of one another. Out of the many balls some were above the liquid surface—while that at the bottom was so deeply in the liquid that it reached the left arm of the U tube and floated to the surface. Since the liquid level was much higher in the left arm than in the right one, when the ball reached the surface it fell. While falling, it turned a paddle wheel, making useful work, and then landed in the right arm, on top of the other balls. Then the new ball at the bottom was pushed into the left arm—and the whole process started again.

   How did Einstein clear up the mist for the mistaken inventor? Answer every objection the inventor might have. Show that constructions based on the method described above are never functional.

   -3mm

   (Merse Előd Gáspár)

2. Far from the stars, in empty space a spaceship is moving—naturally, at constant speed. To the bow and the stern of the spaceship two gigantic lenses have been fixed, focusing starlight on two water tanks. Because of the Doppler effect, on the bow the light of the "approaching" stars is slightly bluer, while on the stern, the light of the "receding" stars is slightly redder than the average. According to Planck and Einstein, photon energies are proportional to their frequencies, thus the water tank on the bow absorbs more heat per unit time than its counterpart on the stern. Thus water will be somewhat warmer in the front tank than in the back one. A heat engine installed between the two tanks is used to heat the captain's sauna and feed the cooling fan of the Great Computer. The method provides the crew with a comfortable source of energy forever.

   a) Can the system work? Is it not in contradiction with the second law of thermodynamics (as it uses only one heat reservoir, stellar radiation as its source of energy)?

   b) Some say that the spaceship will sooner or later come to a halt. What physical facts do these "Some" refer to? And, after all, what does "come to a halt" mean? Relative to what? What does Einstein have to say on the matter?

   -6mm

   (Gyula Dávid)

3. One evening the young Albert was sitting at the kitchen table, fighting with his obligatory daily dose of oranges. To tell the truth, he was just playing with the oranges instead of eating them—for he happened to find a perfectly hemispherical half orange, which he repeatedly picked up, turned edgeways, and with an initial push set into rolling on the table. Absorbed in the fascinating dance of the orange, Albert started pondering whether it was possible, in principle, to set the orange in motion so that it would trace out a straight or a circular path.

   a) Try to answer the above questions.

   Needless to say, the freshly washed orange usually followed a meandering path, tracing out beautiful figures in the very fine flour layer covering the table. Gung-ho, Albert gobbled up the oranges, and spent the rest of the evening with the mathematical analysis of the curves.

   b) Let us do the same. Do the curves possess a general qualitative property independent of the character of the damping used in the calculations?

   (*Hint:* The mass distribution of the oranges is assumed to be uniform.)

   -6mm

   (Péter Rakyta)

4. As part of 'Thrown-Up Stone' project, the American Gun Club has finally made its spaceship gun, and named it after Jules Verne. (The gigantic earthwork had some real spin-offs: the earth's rotation was stopped, and its atmosphere vanished. But something for something: J. T. Maston has one less parameter to take care of in his calculations.) The gun launches its projectile at the second cosmic speed, which then leaves the earth in the radial direction. Not long before the countdown, the Gun Club learns that 10,000 kilometers away their German rival has produced an identical gun. Moreover, the Germans plan to launch three projectiles at one-hour intervals– spaceships 'Zero Stone', 'One Stone', and 'Two Stone'. The second projectile is to be launched simultaneously with the Florida spaceship 'Verne'. Needless to say, the constructors equip the spaceships with radars following the motion of the others.

   D-day arrives. The crew of spaceship 'One Stone' (Ein Stein) is awaiting the unprecedented space voyage in excitement. Bang. . . an enormous jerk. . . apparently the shock absorbers could have done better. g-LOC sets in. An hour and a half later the crew members come by, only to find out what is meant by amnesia: they have no idea who and where they are. Puzzled, they float in the cabin. Their only source of information is the radar, showing the position of the three other metallic objects relative to themselves. The crew members slowly start to remember the most important thing: they are all physicists who learned Newton's laws and the Physicists' March (`http://mafihe.hu/indulo.html#english`). From the fact that they are floating they draw the conclusion: they are in an inertial frame. Next they determine the force law governing the acceleration of the three other objects relative to them. Let us do the same.

-5mm

(Gyula Dávid)

5. While working out the special theory of relativity Einstein was pondering a great deal about the physical and philosophical reasons behind the privileged status of rectilinear motion with respect to accelerated—e.g. circular—motion. Let us do the same.

   Calculate how much time a spider needs to reach a point at 3.14 meters and come back, if it is to arrive at the starting point with zero speed. How much time does it need to run along the circumference of a circle two meters across, if starts and arrives at the same point with zero initial and final speed?

   The friction coefficient between the spider and the (flat) surface is 0.4; there is no bound imposed on the spider's speed.

   -4mm

   (Zsolt Bihary)

6. In his first article, published in 1901, Einstein studied phenomena occurring in liquids. Let us do the same.

   A ray of laser light (e.g., a laser pointer) is shone perpendicularly on one of the planar side walls of a transparent tank filled with water. The source of light, initially below the water level, is moved upward. What is observed on the paper screen behind the tank? How does the picture change if the tank is filled with some other liquid?

   (Zsuzsanna Rajkovits and Péter Kenesei)

7. In his first scientific works Einstein studied surface tension in liquids. Let us do the same.

   It is well known that if a positive bulge or a negative dip is created on a part of a water surface, then—after a long time—a positive (bulging) soliton wave will usually result in both cases. Although tricky, it is also possible to create a dip-shaped traveling soliton wave. What conditions must be met for this to happen (e.g., for the interfaces between water and air / salt water and freshwater, for the depth of water)?

   -4mm

   (Imre Jánosi)

8. Anecdotes have it that already in his early childhood Einstein found a deep interest in playing with magnets. Let us do the same.

   Two parallelly oriented permanent magnets are placed along a straight line coaxially, so that their separation is large compared to their size. The magnets are initially at rest. Study their motion. How can the strength of the magnets be determined from time and distance measurements?

   Friction, air resistance and all other damping effects should be neglected. Study the general case, too, in which the magnets are coaxial neither with one another nor with the line connecting their centers.

   (József Cserti and Gyula Dávid)

9. Einstein has taught us that in a sufficiently strong gravitational field even light follows a circular path. However, not only gravitation can curb the trajectory of light: Fermat's principle can do it as well.

   Consider an optical medium, whose index of refraction n depends only on the distance r from the origin. How should the function $n(r)$ be chosen if light moves in an elliptical path, and

   a) the origin is at the center of the ellipse;

   b) the origin is at the focus of the ellipse?

   Compared to the ellipse, what is the maximum extent of the region characterized by the refractive index function $n(r)$?

   Hint: With a suitable (cyclic) choice of the coordinate it is straightforward to find the first integral of the differential equation of the path.

   -4mm

   (Gyula Dávid)

10. Show that classical electrodynamics is not covariant nonrelativistically.

Hint: First determine the Galilei transformation formulas of the electric field and the magnetic induction assuming the covariance of the Lorentz force law. Verify that the Maxwell equations for $\mathbf{E}$ and $\mathbf{B}$ are covariant under Galilei transformations. Then show that the Maxwell equations for the electric displacement vector and the magnetic field cannot be covariant if the covariance of the constitutive relations $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = (1/\mu)\mathbf{B}$ is assumed.

Next, sketch how you would search for transformations under which classical electrodynamics is covariant. How could Poincaré arrive at the Lorentz transformation equations?

-4mm

(Previously assigned problem, Tamás Fülöp, 1993)

11. It is well known that Einstein pondered a great deal about moving mirrors. According to some unconfirmed sources once, standing by the rails, he stared for several long seconds at a lady making up in her tiny hand mirror in the train zooming past him. Let us do the same. (No, you do not have to get on a train to fix your make-up, you have to ponder about moving mirrors.)

Consider a parabolic mirror, moving at a velocity $\mathbf{v}$ relative to the observer. An object—stationary in the mirror's reference frame—is placed at a distance $\mathbf{r}$ from it. Study the images formed by the mirror. Trace individual light rays, and calculate the position, size, and orientation of the created image—in Einstein's reference frame.

-3mm

(József Cserti and Gyula Dávid)

12. Everybody knows that a point-like unit charge sitting at $\boldsymbol{x} = \boldsymbol{x}_0$ can be described by an electrostatic potential $\phi(\boldsymbol{x}) = 1/|\boldsymbol{x} - \boldsymbol{x}_0|$, for which

$$\Delta\phi(\boldsymbol{x}) = -4\pi\,\delta(\boldsymbol{x} - \boldsymbol{x}_0)\,. \tag{1}$$

In other words, the charge distribution $\tilde{\rho}(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$ is almost everywhere zero and is concentrated on a point so that the total charge $\int d^3x\,\tilde{\rho}(\boldsymbol{x})$ is unity. For that the formula should indeed apply to a charge sitting in a specific point of $\mathbb{R}^3$, the components of the vector $\boldsymbol{x}_0 = (x_0, y_0, z_0)$ have to be *real*. However, when the 3-dimensional Laplacian is applied to the complex valued function

$$\phi(\boldsymbol{x}) = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \tag{2}$$

of any *complex* vector $\boldsymbol{x}_0$, the result is almost everywhere zero again. (Prove this statement.) Note that $\boldsymbol{x}$ is always a real vector, only $\boldsymbol{x}_0$ is allowed to take complex values. Moreover, a choice of square root is obviously necessary, but suppose the choice has been made once and for all.

Now, a complex valued electrostatic potential, such as (2), clearly does not make physical sense. (Why?) However, in order to end up with something real and physical, one might try to define

$$\varphi(\boldsymbol{x}) = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} + \frac{1}{\sqrt{(x-\bar{x}_0)^2 + (y-\bar{y}_0)^2 + (z-\bar{z}_0)^2}}\,, \tag{3}$$

where the bar means complex conjugation. Clearly, the above function is real and its Laplacian is almost everywhere zero for any choice of the complex 3-vector $\boldsymbol{x}_0$. Thus (3) may be used to describe a physical charge distribution, with the density $\rho(\boldsymbol{x})$ defined by the equation

$$\Delta\varphi(\boldsymbol{x}) = -4\pi\,\rho(\boldsymbol{x})\,. \tag{4}$$

What is $\rho(\boldsymbol{x})$ like? What is its total charge? What is its support, i.e. at what points is the charge distribution different from zero? How is your answer reconciled with the straightforward guesses: "two point charges sitting at complex locations" or "complex dipole"?

First answer these questions for $\boldsymbol{x}_0 = (i, 0, 0)$. Second, what happens if the first component is not purely imaginary, but has a real part, too? What if all components are arbitrary complex numbers?

What happens to the $SO(3)$ symmetry of $\mathbb{R}^3$? Using $SO(3)$ transformations, a real 3-vector can always be rotated so that it should lie along a prescribed axis. What can be said in the same spirit about a complex 3-vector if $SO(3)$ transformations are allowed? Two point charges at real locations $(\pm x_0, 0, 0)$ are left invariant by the $SO(2)$ subgroup of rotations around the x-axis. What is the stabilizing subgroup of our charge distribution $\rho(\boldsymbol{x})$?

The group $SO(3)$ is understood above as the usual rotation group. How do the answers change if it is replaced by $SO(3, \mathbb{C})$, that is by the group of complex 3×3 matrices A for which $AA^T = 1$? (T stands for matrix transposition.) Which is the "true" symmetry group of our problem, $SO(3)$, or $SO(3, \mathbb{C})$?

And finally, what does this problem have to do with Einstein?

Hint: How are $SO(3, \mathbb{C})$ and the Lorentz group $SO(3, 1)$ related?

-5mm

(Dániel Nógrádi)

13. In his famous 1905 article Einstein disclosed his views "On the Electrodynamics of Moving Bodies". Let us do the same. Since the title has already been taken, let us ponder "On the Electrodynamics of Rotating Bodies". Transform the (nonrelativistic) Maxwell equations into a coordinate system rotating at an angular speed $\mathbf{\Omega}$. (*Hint:* Use rationalized cgs units.) Write down the equations in vector form.

    What is the form of the continuity equation describing mass conservation? What form does the equation of motion take for a point charge q of mass m? How can one introduce the usual vector and scalar potentials? What is the form of the wave equation for field quantities and for potentials? Find the dispersion relation of electromagnetic waves propagating in charge- and current-free regions.

    Examine whether the rotating-frame equations have solutions corresponding to a pair of particles of mass M and m, of charge $+e$ and $-e$, both at rest, separated by a distance R. If there is no such solution, what mathematical approximations must be made so that such a solution should exist? What is the physical meaning of these approximations? How should the angular speed Ω of the rotation be chosen?

    -4mm

    (Gyula Dávid)

14. When he read the vulgarized versions of his own theories in the newspapers, Einstein was sometimes tearing out his hair—so familiar to all of us from hundreds of photos. Let us do the same. And then let us take a closer look at the bathtub, filled with water of temperature T, with a hair of length L floating on the surface. The hair, which stays entirely on the surface throughout the process, is unstretchable, but perfectly flexible—it can even cross itself. Its linear energy density is proportional to the square of the local curvature. The angle between the tangents drawn to the two endpoints of the hair is denoted by φ. Determine the probability density function for φ.

    (Győző Egri and Bálint Tóth)

15. By working out the theory of Brownian motion, Einstein laid the microscopic foundations of the Fourier equation of diffusion, according to which the time derivative of the concentration $n(\mathbf{r}, t)$ is proportional to Δn, where $\Delta = \nabla^2$ is the Laplacian.

    But can we really trust this equation? Consider a metal plate, in which the concentration of the solved additive changes linearly between the two boundary surfaces. Because of the linearity, $\Delta n = 0$. Does then such an inhomogeneity persist in the plate forever?

    -3mm

    (Tamás Geszti)

16. The streets of a neighborhood form a practically infinite square lattice. Einstein's 1905 explanation for the Brownian motion applies equally for drunken people on this lattice. When drunken people get to crossroads, they choose each of the four possible direction (forward, left, backward, right) with the same probability.

    a) Suppose that the street section between two adjacent crossroads is crossed. A pub is located at one end of the closed section. A physicist, living at the other endpoint of the closed section, staggers out of the pub. Find the probability that he gets home before getting back to the same pub again.

    b) What is the answer if our physicist is so pissed that he decides not to walk but to take his car? His algorithm is the same as before, but the closed section is no longer a problem for his SUV. (Otherwise, he subconsciously respects the highway code.) All the streets are two-way. However, the steering wheel is stuck, so the car can only go straight or turn to the left. (Initially, it can start in any direction.)

    -3mm

    (Merse Előd Gáspár)

17. Two tachions move at a speed $2c$ relative to the observer. Determine the magnitude and direction of their speed relative to each other, if they move

    a) in antiparallel directions

    b) in perpendicular directions

    in the observer's frame.

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    (Previously assigned problem, Gyula Dávid, 1992)

18. Ever since it was set in orbit in 2315, the Einstein Galactic Space Station has been moving at constant speed along a circular path that is 100 lightyears across and reaches out to the Solar System. The crew have been feeling an inertial (centrifugal) acceleration of $2g = 20\ \mathrm{m/s^2} = 2\ \mathrm{lightyears/year^2}$, as approved by the Bioadaptation Office.

    In which years does the space station zoom past the Solar System? How much older do the crew members get in a cycle?

    The space station is kept in its circular orbit by proton-antiproton rockets (PAPRs). (PAPRs are practically lossless jet engines from the late 23rd century, powered by collimated photons created in the annihilation of separately stored proton and antiproton plasmas.) The fuel to be used is transported to the space station once each cycle, by a PAPR-powered robot spacecraft moving along an approximately straight line at a constant acceleration of 200 g. The mass of the empty robot spacecraft is negligible compared to that of the transported fuel.

How much fuel has to be transported to the space station each cycle if the useful mass of the space station is 10,000 tons? When launched from the earth, how much fuel should the robot spacecraft contain? When should it be launched relative to the time of the closest encounter between the space station and the earth?

(Zsolt Bihary)

19. A laser is accelerated along its axis. How are its properties modified compared to those of an identical but stationary laser? (Note: besides its color, light has several other properties.) The laser can be modeled as a rod-shaped solid with a mirror at each end. Can anything depend on whether the acceleration is due to a surface force or a bulk force? . . . Or a gravitational force?

-3mm

(Titusz Fehér)

20. On several occasions Einstein recalled that his later interest in the problems of relativity rooted in a childhood thought experiment: he tried to imagine what he would see if he was keeping pace with light. Would he see co-moving light waves? Or stationary light waves? Thanks to his later efforts we now know that this cannot happen in empty space: nobody can catch up with light propagating at a speed c. But what about light propagating in a medium of refractive index n, at a speed $u = c/n$? Cherenkov's electrons are known to overtake light in water. Let us do the same.

Write down the source-free Maxwell equations in a medium of infinite extent, of dielectric constant ϵ, and magnetic permeability μ, moving at a constant speed $\mathbf{V}$ relative to the observer. Find the plane wave solutions of wave vector $\mathbf{k}$ and frequency ω. Derive their dispersion relation $\omega(\mathbf{k}, \mathbf{V})$. Identify *absolute standing waves*, i.e. wave solutions with $\omega = 0$. Examine the possible polarization properties of these waves, and the possible configurations of the vectors $\mathbf{V}$, $\mathbf{k}$, $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$. Discuss the results for various values of u (the speed of light in the medium) and V (the speed of the medium relative to the observer). How can one observe these *absolute standing waves*? What would Einstein "see" if he was running (or standing) next to such a wave? *Hint:* Use rationalized cgs units.)

(Gyula Dávid)

21. Muons decay into electron–neutrino–antineutrino trios. How does the momentum distribution of the created particles depend on the momentum p_0 of the incoming unpolarized muon? The question of muon decay is treated within the framework of the standard model of particle physics: according to it, the momentum distribution of the (approximately zero-mass) electron created in the isotropic decay of an unpolarized muon is given by $f(x) = 16x^2(3 - 4x)$, where $x = p_{\text{electron}}/m_{\text{muon}}$, with the constraint $0 < x < 0.5$.

Examine also the following generalization of the question. The momentum of the incoming particle is p_0, its mass is m_0, while the mass of the particle created in the decay process (in addition to the neutrino-antineutrino pair) is p and its mass is m (no longer negligible). The distribution function of the outgoing particle in the center-of-momentum frame, $f(p_{\text{CM}}, \theta)$, is known, and it is not necessarily isotropic. Determine the distribution function of the outgoing particle in the laboratory frame, $f(p, \theta)$.

-3mm

(Sándor Katz)

22. Examine the similarities and differences of the thermodynamic changes taking place in an expanding photon gas under the following conditions.

a) Photon gas in a box: Electromagnetic radiation is confined to a box with reflecting walls. The walls are slowly moved outward, so the volume is increased, and the electromagnetic waves reflected from the walls undergo a Doppler shift—thus the spectrum is slowly changed. If the distribution was originally Planckian, so will be the new one—at a different temperature. This gives the description of the adiabatic expansion of a photon gas in a box. Examine the details of the process, and determine the relation between the volume and temperature of the photon gas.

b) Photon gas expanding freely in space: A well-known example for this situation is the electromagnetic radiation in equilibrium with the solar photosphere (about 6000 K), which undergoes substantial expansion as it leaves the sun's surface. What is the temperature of the radiation when it reaches the earth? What would be the earth's average temperature if it had no atmosphere? (The greenhouse effect of the atmosphere makes things even more complex.)

c) Cosmic background radiation: Cosmologists tell us that a similar process takes place in the expanding Universe. The photon gas, relic of the Big Bang, that was once in thermal equilibrium with all other forms of matter, became decoupled, and has been in continuous expansion and cooling ever since. At first sight this situation is similar to b). Nonetheless the same cosmologists confirm that the radiation spectrum is Planckian at all times, with a steadily decreasing temperature—and they speak of an adiabatic expansion of the photon gas, just like in a). In this case, however, there is no reflection process behind the Doppler shift of each wave. How is it possible that the photon gas keeps on expanding adiabatically, through states characterized by well defined temperatures? With what kind of matter is the radiation in thermal equilibrium?

As a good student of Sherlock Holmes, find some underlying symmetry. What would change if photons had a finite—i.e., nonzero—mass?

-3mm

(Previously assigned problem, Gyula Dávid and Péter Hantz, 1996)

23. Einstein's special theory of relativity is based on a 4-dimensional spacetime, described at each point by a locally Minkowskian metric, but whose global topology is left unspecified. Imagine a spacetime in which one of the spatial coordinates is cyclic, i.e., it has a global topology of a closed cylinder ($S^1 \times R^3$), and re-examine the usual twin paradox.

    Let two twins be A and B. Twin A is waiting at the rail station, while B zooms past, sitting on board the Relativistic Rail, which is a superfast train continuously circling the universe. The Relativistic Rail travels along a straight line with a large constant velocity. At the instant of their meeting, A and B are of the same age.

    a) During the trip, B is carefully observing the great clock tower found in the close vicinity of the station and compares its time to that shown on her own watch. Does she observe the station clock run faster or slower than her own? What does the answer depend on?

    b) Which of the two twins will be older at the time they meet again at the rail station? What is the origin of the (a)symmetry?

    c) Assume that the Relativistic Rail never stops at the station, but travels on with an unchanged velocity. What is the intensity of the sound of the engine, as heard by A and B, as a function of time and frequency?

    (Assume that the length of the railroad is $L = 1$ lightyear, the engine velocity is $v = 0.99\,c$, and the engine emits a constant I intensity sound at $f = 1000$ Hz in the comoving reference frame. The speed of sound is $10^{-6}c$. The twins' eyes and ears are perfect.)

    -3mm

(Zoltán Haiman and Bence Kocsis)

24. According to the special theory of relativity, time passes more slowly in a moving spaceship. On the other hand, according to the general theory of relativity, gravitational fields slow down the course of time—that is, clocks tick faster in rockets quitting the earth's gravitational well. Stealth Spaceship Richard Feynman was assigned the following mission: it is to leave the terrestrial base in the radial direction, and return to it after precisely one terrestrial day. The spaceship's clocks must show the physically possible largest lapse of time. How should the captain operate the rockets to accomplish the mission?

    (The rocket moves radially at all times. The earth's revolution and the perturbing effects of the atmosphere should be neglected.)

    -4mm

(Previously assigned problem, Péter Gnädig, 1996)

25. The protagonist of Frederick Pohl's sci-fi novel *Gateway* accidentally had pushed his girlfriend into a black hole. He has had a life-long remorse for this, as he knows that time slows down in the vicinity of a black hole, and his girlfriend's indignant expression will be visible forever. In the second volume the protagonist gets rich, makes new plans—and will probably go after the girl in the forthcoming third volume. He has plenty of time, since the last ten years passed as a mere half hour for her. Should the publisher speed things up, or does the protagonist indeed have an eternity at his hands?

    PS: Since the original assignment of the problem, the third volume came out. The reader learns that the Aliens pulled the lady out of the black hole with the help of a "gravitational can opener". Nonetheless, not even this fortunate turn of the events can change the physical content of the problem assigned 13 years ago.

(Previously assigned problem, Gyula Dávid, 1992)

26. Examine "Einstein's daisy". According to the general theory of relativity, the elliptical planetary orbits are not perfectly closed. In extreme mass ratio compact binary systems (e.g., a large black hole plus a neutron star) closed orbits may nonetheless exist for suitably chosen initial conditions.

    How should the initial conditions be chosen so that in a sufficiently but not exceedingly long period of time the neutron star should trace out a figure similar to the petals of a daisy? Describe the associated gravitational waves.

(Szabolcs Márka)

27. Examine Einstein's cannon. Assume that for some strange reason a small fraction (say, 10^{-11}) of the mass of a neutron star is asymmetrically ejected into space at high velocity ($c/2$). Calculate the time evolution of the mass quadrupole moment. Determine the intensity of the emitted gravitational waves (if any).

(Szabolcs Márka)

28. Recent observations showed a slowly increasing blueshift in the radio signals of two space probes currently at approximately 20 astronomical units from the sun and receding in different directions. Analysis of the data of the past few years has shown that this corresponds to a constant sunward acceleration $10^{-10}\,\mathrm{m/s^2}$ of unknown origin. Several researchers have proposed that the phenomenon is caused by the expansion of the universe, since the product of the Hubble constant and the speed of light is more or less of the same value. Verify or refute this hypothesis through precise calculations.

(Gyula Bene)

29. Suppose that beyond our known world there is another, similar world, the 'Other world". The two are connected by a wormhole, through which brave astronauts can get to the Other world safe and sound. A possible solution for this is the following spacetime.

Let the coordinates be (t, l, θ, ϕ), with $l > 0$ in our known word and $l < 0$ in the other one, and $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$ the usual spherical angles. Let the square of the infinitesimal interval be given by

$$ds^2 = e^{2\Phi(l)}dt^2 + dl^2 + r(l)^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

where $\Phi(l) = -2M/|l|$, and $r = |l| - M\ln(|l|/r_0)$. Here $M > 0$ is the mass of the wormhole, and $r_0 > 0$ is the "radius" of its throat. This spacetime is indeed asymptotically Minkowskian in the $|l| \to \infty$ limit.

a) How should M and r_0 be chosen so that the observer descending slowly and radially into the wormhole should not feel more than the terrestrial acceleration g?

b) How should M and r_0 be chosen so that the tidal force experienced between her feet and head by a cautious, therefore slowly descending observer of 180 cm should not exceed $1g$?

c) How should M and r_0 be chosen so that the tidal force experienced between her feet and head by a dauntless observer of 180 cm should not exceed $1g$, when she plunges head first radially into the wormhole?

(Bence Kocsis)

30. Examine the character of the matter sustaining the wormhole in problem 29.

a) Write down the t-t and r-r components of the Einstein equations, and determine the spatial dependence of density and pressure.

b) What density and pressure are observed by an observer moving radially at a speed approaching that of light? Why is this important?

(Bence Kocsis)

31. Two well-known spherically symmetric solutions of the Einstein equations are the Schwarzschild black hole of mass M and the anti-de Sitter spacetime. These solutions are derived in mutually exclusive cases: the Schwarzschild black hole solution does not assume a cosmological constant, and the cosmological solution does not assume a central singularity.

a) Derive the general static and spherically symmetric "vacuum solution", where both $M \neq 0$ and $\Lambda \neq 0$.

b) How does the Schwarzschild radius of the black hole's event horizon change compared to the standard solution? Numerically verify that the observed value of the cosmological constant in the present-day universe does not significantly modify the horizon radius of solar-mass black holes. How large should a black hole be for that this should not be the case?

c) Assume that black holes existed in the inflationary era shortly after the Big Bang. How is the answer to question b) modified for these primordial black holes?

-2mm

(Bence Kocsis)

32. In Woody Allen's celebrated film *Annie Hall*, the young Alvy Singer (depicting Woody Allen himself) develops a childhood depression upon learning that according to Einstein's theory the Universe is expanding. He probably fears that also Brooklyn is expanding—and so is himself. Cosmologists usually reassure him by saying that only "sufficiently large" objects are subject to cosmic expansion, small ones held together by local interactions (e.g., atoms, Alvy, Brooklyn, the Earth, the Solar System, the Milky Way, etc.) are not. But is this correct indeed? What is meant by "sufficiently large" in this context? Examine the question within the framework of classical mechanics.

In models of the expanding Universe a "comoving" cosmological coordinate system is usually defined, in which the position vectors $\mathbf{R}$ of objects "at rest" (e.g., galaxies) are constant in time. Physical displacement vectors $\mathbf{r}(t)$ are related through $\mathbf{r}(t) = a(t)\,\mathbf{R}$, where $a(t)$ is in general a steadily increasing function that describes cosmological expansion (and the origin can be arbitrarily chosen because of homogeneity). Thus the acceleration of a body at physical distance $\mathbf{r}(t)$ from the origin is $\ddot{\mathbf{r}}(t) = \ddot{a}(t)\,\mathbf{R} = (\ddot{a}/a)\,\mathbf{r}(t)$. The above may be interpreted as if a body of mass m located at a distance $r(t)$ from the origin were acted upon, beyond the usual local forces, by an additional "cosmological force" $\mathbf{F}_{\mathrm{cosm}} = m(\ddot{a}/a)\,\mathbf{r}$.

a) Taking into account the above cosmological force, study the Keplerian orbits of a body of mass m in the gravitational field of a much larger mass M located at the origin. Separate the equations of motion in the usual two-dimensional polar coordinate system, and find the differential equations for the radial physical coordinate $r(t)$ and the "radial cosmological coordinate" $R(t) = r(t)/a(t)$.

b) Recent measurements show an accelerating expansion of the Universe, that is, the expansion function $a(t)$ is of the form $Ce^{\alpha t}$. Place the central star of the above one-planet solar system into the origin, and study the behavior of the planet in the background of the exponentially expanding universe. Discuss the solutions in terms of the expansion parameter α, the stellar mass M, and all other significant parameters. What is the planet's fate in the $t \to \infty$ limit? (Note: Analytical studies, and not numerical solutions are expected for this question.)

c) Perform numerical studies to determine the temporal variation of the orbital radius of a planet initially on a circular orbit of radius r_0 for other expansion functions (e.g., $a(t) \propto t^{2/3}$, t^2 or $1 + t^2 \tanh kt$). What can be said about the planet's fate? How can you reassure Alvy?
-2mm

(Gyula Dávid, from R.H.P.'s idea)

33. Ultra high energy cosmic rays (UHECR) occasionally reach the earth. So far 14 particles (most probably protons) have been detected whose energy exceeded 10^{20}eV. The all-time high particle energy, $\approx 3 \times 10^{20}$ eV was recorded by the detector "Fly's Eye".

A possible scenario for the creation of such particles is this: Along with the cosmic microwave background, the Big Bang theory also predicts the existence of a cosmic neutrino background (approximately 56 neutinos per cc). These neutrinos are essentially at rest. It is suspected that numerous UHE neutrinos reach the earth as well as UHE protons. When these neutrinos hit their stationary colleagues, Z-bosons may be produced – as long as the rest mass of the stationary neutrino is sufficiently large and the cosmic neutrinos are really energetic. The Z-bosons then disintegrate into various particles, among other protons. These protons (and antiprotons) might then be the detected UHECR-particles.

Suppose that the proton produced in the Z-decay takes about $1/100$ of the Z particle's energy (in the Z rest frame). Determine the neutrino mass if the all-time champion ($\approx 3 \times 10^{20}$ eV) proton had been created this way, supposing the outgoing proton was moving in the same direction as the incoming neutrino. What is the answer if one takes into account the angular distribution of the protons, $p \propto 1 + \cos^2 \theta$, where θ is the angle between the directions of motion of the outgoing proton and the incoming neutrino, measured in the Z rest frame?

The above determination of the neutrino mass is based on the idea that the incoming neutrino energy necessary for Z production depends on the neutrino mass. This rang the bell for Prof. Finga Reen Avery 3.14 – fractional descendant of the great fuzzycist, Finga Reen Avery 0 –, who immediately proposed the following experiment:

Let two neutrino beams collide. The energy of one beam is kept fixed (around the Z mass), and the energy of the other is tuned so that Z bosons are produced in the collision. As demonstrated above, the neutrino mass can be calculated from the two beam energies. Should we recommend that the experiment be realized (supposing that the technical difficulties of producing beams of sufficient luminosity around the Z energy have been overcome)?
-2mm

(Previously assigned problem, Sándor Katz, 2001)

34. Consider the following modification of a dust-dominated, expanding, flat Friedmann-Robertson-Walker universe, and show that it is an exact solution to the Einstein equations.

Disjoint spheres of different radii are removed from the matter uniformly distributed throughout space, and replaced by point masses in the centers of the spheres. The mass of each point is equal to that of matter within the (corresponding) removed sphere. Inside the spheres, the spacetime metrics is Schwartschildian. What is light propagation like in such a universe?
-3mm

(Gyula Bene)

35. Einstein was awarded the Nobel prize for proving the existence of photons through the interpretation of the Lenard law of photoelectric emission. Einstein was not satisfied with this proof, and kept on searching for more convincing evidence for the existence of photons—which he eventually found in the laws of spontaneous and stimulated emission. What was wrong with his first, Nobel prizewinning proof? Is it possible to interpret the behavior observed in the experiments quantum mechanically, but without photons?

(Tamás Geszti)

36. In recent Bose-Einstein condensation experiments atoms (which are bosons) are trapped in (optically or magnetically created) potentials of the form

$$V(x, y, z) = \frac{1}{2} m \left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right),$$

where ω_x, ω_y and ω_z are the so-called "trap frequencies", and m is the mass of one atom. Consider N noninteracting atoms placed in this potential. What is the critical temperature T_c of Bose-Einstein condensation? How does the number of atoms within the condensate change for temperatures $T < T_c$? Estimate the critical temperature T_c for ^{23}Na isotopes, if $N \approx 10^9$, and if typical trap frequencies are $\omega \approx (2\pi) \cdot 200$ Hz.
-3mm

(András Csordás)

37. Spin-orbit interaction is known to be a typical relativistic effect. Consider an electron moving along an infinitesimally thin circular ring of radius a in the xy plane, as described by the Hamiltonian

$$H = \frac{L_z^2}{2ma^2} + \frac{\beta}{a}(\mathbf{e}_\varphi \cdot \mathbf{S})L_z + i\frac{\hbar\beta}{2a}(\mathbf{e}_r \cdot \mathbf{S}).$$

Here $\mathbf{S}$ is the electron spin operator, L_z is the operator of the z component of the orbital angular momentum, $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ are the radial and azimuthal unit vectors of the two-dimensional polar coordinate system, and β is a real parameter.

Apply a uniform magnetic field B, perpendicular to the plane of the ring. Determine the energy eigenvalues and energy eigenstates of the obtained system, as well as the current of each eigenstate.

(András Pályi)

38. In textbooks on relativistic quantum mechanics one can often read things like this: "The Dirac operator commutes with the angular momentum component operators if and only if the spatial part of the vector potential is zero, and the temporal part is spherically symmetric." That is, the spherical symmetry of the problem is expressed in terms of the properties of the vector potential. Naturally, this is not gauge invariant (since the gauge is fixed in the above assertion; usually the Lorenz gauge is adopted, with the auxiliary condition that the potentials should vanish at infinity.)

Find a gauge invariant condition (that is one expressed in terms of the field strengths) for the spherical symmetry of the Dirac problem—i.e., for the commutation of the angular momentum components with the Dirac operator.

(András László)

39. Consider a Michelson interferometer with one photon. At the end of one arm an oscillating mirror is located. The frequency of the photon is ω_c, that of the mirror is ω_n. The mirror is small enough to be treated quantum mechanically, and its oscillation amplitude is negligible in second order compared to the length L of the arms.

a) Write down the Hamiltonian of the system.

b) In what state is the system found after time t if the initial condition is chosen as

$$\mid \Psi_0\rangle = \frac{1}{\sqrt{2}}\Big(\mid 0\rangle_A \mid 1\rangle_B + \mid 1\rangle_A \mid 0\rangle_B\Big)|0\rangle_m.$$

(A and B are the state indices of the two arms, while m is that of the mirror.)
c) How does the visibility of the interference fringes change with time?

(József Zsolt Bernád)

40. Examine a harmonic oscillator with velocity-proportional damping. Quantize this system, and interpret the result.

Next, place an undamped oscillator into a thermostat (which can be modeled by a large number of tiny oscillators interacting with the system under examination). Quantize this system, too, and compare the result with that of the previous question, as well as with the results of Brownian motion.

(József Zsolt Bernád)

41. Calculate the attractive force between two square-shaped gold plates of 1 cm^2, 10^{-6} cm apart, making recourse to

a) the Casimir effect;

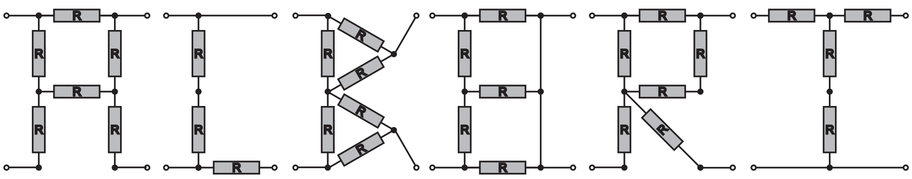
b) Van der Waals forces.

How do these forces depend on separation and material properties (i.e., plates made of some other metal)?

(István Csabai)

42. Each link of a chain of length N is chosen randomly from the four-terminal elements below. What is the expected value of the net resistance as a function of N? Examine the limit of large N.

Note: The four-terminal elements are not rotated, and the net resistance is measured between the two free terminals of the first element.
5mm



(Merse Elő̋d Gáspár)