THE 35th
—7th INTERNATIONAL—
RUDOLF ORTVAY
PROBLEM SOLVING CONTEST IN PHYSICS
2004

The Physics Students' Association of Eötvös University, Budapest, the Roland Eötvös Physical Society and the Hungarian Association of Physics Students proudly announce the **35th** – and for the seventh time **international** –

Rudolf Ortvay Problem Solving Contest in Physics,
from 29 October 2004, through 8 November 2004.

Every university student from any country can participate in the Ortvay Contest. PhD students compete in a separate category. The contest is for individuals: solutions sent by groups of students are not accepted. The name, the university, the major, and the university year should be indicated on the solutions. Pseudonyms and passwords cannot be used: each contestant has to use his/her own name. The problems can be **downloaded** from the webpages of the Ortvay Contest

http://ortvay.elte.hu/

in Hungarian and English languages, in html, LaTeX pdf, and Postscript formats, from **12 o'clock (Central European Time, 11:00 GMT), Friday, 29 October 2004**. The problems will also be distributed by local organizers at many universities outside of Hungary.
Despite all the efforts of the organizers, it may happen that some unclear points or misprintz stay in the text. Therefore it is very useful to visit the webpage of the contest from time to time, as the corrections and/or modifications will appear there.
Each contestant can send solutions for up to 10 problems. For the solution of each problem 100 points can be given. *Each problem should be presented on (a) separate A4, or letter-sized sheet(s). The contestants are kindly asked to use only one side of each sheet. Solutions written by pencil or written on thin copy-paper will not be accepted as these cannot be faxed to the referees.*
Any kind of reference material may be consulted; textbooks and articles of journals can be cited. Computer programs appended to the solutions should be accompanied by detailed descriptions (what computer language it has been written in, how to use the program, which parameters can be set, what notations are used, how to interpret the output figures and graphs, etc.) They can be enclosed on floppy disks, or sent via email to the addresses below. Solutions can be sent by mail, fax, or email (in LaTeX, TeX or Postscript formats—or, if they contain no formulae, in normal electronic mail). Contestants are asked not to use very special LaTeX style files unless included in the sent file(s).

Postal Address:

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Deadline for sending the solutions: 12 o'clock CET (11:00 GMT), 8 November 2004.

Contestants are asked to fill in the form available on our webpage after posting their solutions. It will be used for identification of contestants and their solutions. **Without filling in the form, the organizers cannot accept the solutions! The form is available only on 8th and 9th November.**

The contest will be evaluated separately for each university year, according to the total number of points. The referees reserve the right to withhold, to multiply or to share some prizes. Beyond the money prizes given for the first, second, and third places, honorable mentions and special prizes for the outstanding solutions of individual problems can be awarded. This is why it is worthwhile sending even one or two solutions.

The announcement of the results will take place on 16 December, 2004. The detailed results will be available on the webpage of the contest thereafter. Certificates and money prizes will be sent by mail. We plan to publish the assigned problems and their solutions in English language—to which the contribution of the most successful participants is kindly asked. The volume is planned to be distributed all over the world with the help of the International Association of Physics Students, as well as the contestants themselves. We hope this will help in making the contest even more international. Wishing a successful contest to all our participants,

the Organizing Committee:

Gyula Dávid, Attila Piróth, József Cserti
(Eötvös University, Budapest, Hungary)

1. Go to the link

   `http://ortvay.elte.hu/2004/budapest.jpg   (1.6 MB)`

   to find a space photography of the inskirts of Budapest, stretching from the southern tip of Szentendre island to the campus of Eötvös University, in Lágymányos.
   a) When (what month, day, hour, and minute) was the photo taken?
   b) Determine the altitude of the clouds.

   (András Pál)

2. From Friday to Friday, with the help of Man Friday, Robinson made a square-shaped pond, to spend his free afternoons swimming. His splashing can only be disturbed by the cannibals who suddenly pop up from the bushes around the pond—and who would certainly appreciate if Robinson was featuring on their menu. Robinson is at the center of the pond when a cannibal appears on the shore. The race for life and death begins—a Robinson steak is at stake! The cannibal cannot swim; his running speed is u. Ashore Robinson runs faster than the cannibal. While Robinson is in the water, the cannibal always runs in the direction that grants him a quicker approach to the point where the line connecting the center of the pond with Robinson meets the shore. Find a lower bound on Robinson's swimming speed that guarantees he can get away.

   Extra question: Does the cannibal have a better strategy than that described above?

   (Szilárd Farkas and Zoltán Zimborás)

3. We wish to illustrate the principle of jet propulsion with a small cart, moving freely on a level surface. The cart has a tank on it that is filled with water, initially up to height H. A horizontal tube of length L is connected to the bottom of the tank, through which the water can flow out "backwards". The cross sectional area of the tube is $1/k$ times that of the tank. The mass of the tank and the cart are negligible compared to that of the water.

   Describe the motion of the cart.

   (Péter Gnädig)

4. Stones are projected vertically upwards from the surface of a very rapidly spinning planet of spherical shape that has no atmosphere. Determine where the stones touch the ground as a function of the latitude of the place of projection and the initial speed of the stone.

   (Gábor Veres)

5. Two solar type stars revolve around one another. Their largest distance is three times as much as the sun-earth distance. The projection ellipses of the orbits on the celestial sphere pass mutually through the center of the other ellipse.

   a) What portion of the year (ie, period of revolution) do the each star spend inside the other star's orbit? (Give a precise numerical answer, preferably without using a computer.)

   b) How does the temperature of a small black body placed in the midpoint of the line connecting the two stars change in the course of the year?

   (Gyula Dávid based on an MIT problem)

6. A practically uniform magnetic field of induction $\mathbf{B}$ is present within a small region of interplanetary space. Also present are two protons: one is initially at rest at the origin, while the other is at position $\mathbf{r}_0$, and has an initial velocity $\mathbf{v}_0$. The vectors $\mathbf{B}$, $\mathbf{r}_0$, and $\mathbf{v}_0$ are mutually perpendicular.

   Find the maximum distance of the two particles during their motion, if only electromagnetic forces act. After how much time will the distance between the protons be the same as initially?

   For what value $v_0 = v_c$ will the distance be a constant as time proceeds (where $v_0 = |\mathbf{v}_0|$)? Examine the special case when v_0 is only slightly different from v_c. Are there any closed orbits?

   Neglect radiation losses, as well as the magnetic field of the moving protons compared to the external field $\mathbf{B}$.

   (Péter Gnädig)

7. The coefficient of restitution describes how a body bounces back from another or from the wall. Its value is given by the ratio of the initial and final magnitudes of the velocity component normal to the plane of rebound. It has been shown experimentally that this value can exceed 1 for some sufficiently soft rebounding surfaces. Model the phenomenon in two dimensions by disks bouncing back from the wall. Explain quantitatively how this is possible without violating the principle of energy conservation.

   (Szabolcs Borsányi)

8. Derive the equation of bicycle wheel tracks.

Consider a rear-wheel drive bicycle of axle distance L, which one rides on a level surface. What is the relation between the tracks traced out by the front and the rear wheel? How should the equation of the track curves be given so that this relation take a simple form?

Apply the result for some simple cases: what track is traced out by the rear wheel if the front wheel follows a) a straight-line path b) a circular path? (The angle formed by the initial direction of the front wheel and the frame of the bike can be arbitrary.) What curve does the rear wheel trace out asymptotically if the front one follows a sinusoidal path? What are the parameters of the curve? Does the axle distance L appear among the parameters?

In a race N cyclists, spaced a distance Δ apart, follow each other with an identical speed v. To make use of the wind shadow, each cyclist follows the track of the rear wheel of the cyclist in front. After a very long straight section the road turns through 90 degrees along a circular arc of radius $R \gg L$. How will the following distances change, if each cyclist keeps on following the track of the rear wheel of the cyclist in front, without changing the speed?

(Merse Előd Gáspár)

9. Three space stations, forming an equilateral triangle on the synchronous orbit, revolve around an atmosphereless, spherical planet. (For simplicity, assume that the radius of the planet, as well as its revolution and rotation periods are the same as those of the earth, and that the central star is similar to the sun.) The following method is proposed for revolutionizing the transportation of freight between space stations: the packets are simply projected "horizontally" (ie along the tangent of the synchronous orbit) from the station; the gravity of the planet will take care of the rest!

a) At what speed should the packet be tossed out if the destination is the station moving in front or behind? How much time does delivery take? Plot the path of the packet in the reference frame co-moving and co-rotating with the station from which the packet was released. (The axes of the reference frame are oriented towards the centre of the planet, the tangent of the circular orbit, and the normal vector of the orbital plane).
b) It occurs to someone that the method can be made more efficient by letting the packets bounce back (once) from the planet. To this end, smooth, horizontal and perfectly elastic "bouncers" have to be installed at certain points of the planet. How do the answers given in a) change?
c) The captain of the station decides that the change proposed in part b) is not worth the fuss, since the maximum attainable savings in the delivery time are about 4%, which does not justify the investment costs of the bouncers. Only one question is left: why does not the planet have an atmosphere?

(Gyula Dávid)

10. Inertial mass is defined by Newton's second law, $\mathbf{F} = m\mathbf{a}$, where m is a constant, independent of the body's position and velocity. However, not only scalar masses can satisfy the linear relationship between the two vectors. What are the consequences of a tensorial choice of the mass m?

a) How are the usual conservation laws modified?

Parts b) and c) below are concerned with motions in a two-dimensional space. Let the mass tensor be isotropic, ie rotationally invariant. (Attention, this is not necessarily a multiple of the unit tensor!)
b) If mass is tensorial, nontrivial stationary states may exist even in dissipative systems without external excitation. Study the motion of a particle with the above-mentioned isotropic mass tensor in a central potential of the form

$$\mathbf{F} = -k\mathbf{x} \cdot |\mathbf{x}|^{b-1} - \eta \mathbf{v}, \quad k, \eta > 0, \quad b \in \mathbb{R}.$$

Under what conditions will the trajectory motion of the particle be bounded? What is the stationary solution?
c) If $N > 1$ identical particles of isotropic mass move in each others' field, the external potential can be omitted:

$$\mathbf{F}_i = -k \sum_{j \neq i}^{N} (\mathbf{x}_i - \mathbf{x}_j) \cdot |\mathbf{x}_i - \mathbf{x}_j|^{b-1} - \eta \mathbf{v}_i, \quad k, \eta > 0, \quad b \in \mathbb{R}.$$

When does a stationary solution exist? What is this solution?

Numerical results leading to analytical considerations are also of interest.

(János Asbóth)

11. A dumb-bell shaped planet revolves around the sun. (The planet can be considered as two mass points, connected by a rigid rod of length d and negligible mass. The motion is planar; the mass of the planet can be neglected compared to that of the sun.) When the planet's orientation is not symmetric with respect to the vector $\mathbf{r}$ drawn from the sun to the planet's center of mass, gravitational forces will tend to turn the planet towards the direction of $\mathbf{r}$. If during its revolution the planet is in its perihelion (aphelion) when a torque in one (in the other) direction is exerted on it, an exchange between orbital and rotational angular momenta can take place.

a) What relation holds between the two types of motion?
b) Can the planet's rotation increase for a long time at the expense of its revolution? If so, under what conditions?
c) When can the planet fall into the sun? (Until the planet reaches the surface of the sun, it is sufficient to calculate gravitational forces to the same order as at the beginning of the motion.)

(Titusz Fehér)

12. A rod pendulum is swinging in a uniform gravity field. At $t = 0$ its angular displacement (from equilibrium) is φ_1, while time T later it is $\varphi_2 = \pi$. Find the lowest-energy solutions, ie discard unnecessary extra turns.

    How does the energy of the pendulum depend on T and the initial angle for asymptotically large values of T?

    (Zoltán Bajnok)

13. Water is rotated at a constant angular velocity ω in a cylindrical tank. A uniform wooden ball of radius R, stationary in the reference frame co-rotating with the water is placed into the water. Where will the ball be after a long period of time? Study the case of small balls, as well as bigger ones, for which $R \gtrsim g/\omega^2$.

    (Bence Kocsis, Győző Egri and Márton Kormos)

14. In a rough approximation, a rapidly spinning neutron star can be modeled by an incompressible fluid held together by its own gravity.

    Show that in this model the star can be spheroidal, as long as the angular speed is not too high. At most how large can its "oblateness" be?

    (Péter Gnädig)

15. Once upon a time, when the weather was particularly windy, Huygens, Fresnel, and Bernoulli went to the prairie to play the drum. The instrument was with Huygens, and from time to time he hit it. Fresnel and Bernoulli were listening from the same distance; Bernoulli against the wind, and Fresnel in the direction of the wind. Wind was blowing horizontally, and its speed varied with the altitude as $v = \alpha z$ (being much smaller than the sound speed at all heights relevant to the problem). What did Fresnel and Bernoulli hear? (The sound emitted by the drum can be approximated by a short delta pulse; the viscosity of air can be neglected.)

    (Bence Kocsis and Győző Egri)

16. The motion of relativistic particles in a static, central Lorentz scalar field $V(r)$ is studied. Neglect the reaction of the particle on the scalar field and the central body producing it. Throughout the problem, do calculations in the inertial frame fixed to the center.

    For parts a) and c), examine the nonrelativistic approximation and its range of validity. What is the physical meaning of the bounds?
    a) Particles move in a circular orbit of radius R. How does the scalar potential $V(r)$ depend on the radius r, if the period of revolution is independent of the radius R of the orbit?
    b) Consider a Keplerian scalar potential $V(r) = -\alpha/r$. Does Kepler's third law hold for relativistic motions in circular orbits?
    c) What is the form of the scalar potential $V(r)$, if relativistic motion is in traditional Keplerian elliptical orbits (ie closed orbits, with the center of attraction in either focus).
    d) What should be the impact parameter of a point particle of energy E, sent towards the center of the scalar potential in part b), if we wish to have a deflection angle that is twice as large as in the nonrelativistic case?

    (Gyula Dávid)

17. Chewbacca looks out of the window of the Millenium Falcon, and suddenly spots out a very rapidly moving meteor, heading for the base of the rebels. Of course, Chewie knows the favorite trick of the Imperial Army: by moving large mirrors, they deceive and draw away the rebel spaceships from their posts. The meteor is enshrouded in a cloud of gas, therefore changes in its shape cannot be observed. How could Chewie verify quickly whether he sees an image reflected from a rapidly moving mirror, or indeed a meteor is heading for the base of the rebels? Does the method work in all cases?

    (Zoltán Zimborás)

18. The formulae of the Doppler effect describe the properties of light emitted by a laser moving at constant velocity relative to the observer. What happens if the *acceleration* of the emitting laser is constant? (Calculate the results as functions of the direction and the magnitude of the acceleration.)

    (Titusz Fehér)

19. The elastic scattering cross section of relativistic electrons by a Coulomb potential is well known from quantum electrodynamics The classical ($\hbar \to 0$) limit of the formula is very simple:

$$\frac{d\sigma(p,\theta)}{d\Omega} = \gamma^2(1 - v^2\sin^2\theta/2)\left.\frac{d\sigma(p,\theta)}{d\Omega}\right|_{\text{Rutherford}},$$

    where p is the three-momentum of the electron, $\gamma = \frac{1}{\sqrt{1-v^2}}$, θ is the scattering angle and v is the electron's speed (all measured in the frame of the Coulomb potential, in $c = 1$ units). Carry out a classical calculation to explain the $\gamma^2(1 - v^2\sin^2\theta/2)$ relativistic correction.

    (Kálmán Szabó)

20. The space travel agency *Black Hole Travels* organizes exotic trips for adventurous billionaires. Travelers can select the black hole they want to visit from a catalog. The participants are then transported into the vicinity of the black hole on a luxurious space ship, which then starts circling the black hole in a circular orbit. Approaching the horizon of the black hole, the travelers may enjoy a fantastic view: the sky closes in, stars turn blue, etc.

    a) How close can they approach the horizon of a static black hole of mass M?

    b) How deeply can they enter into the ergosphere of a Kerr black hole of mass M rotating at maximum angular momentum,

    if the acceleration due to gravity should not exceed the usual terrestrial value, and the tidal forces should not make the travelers feel *too* uncomfortable? Examine the cases $M = 1M_\odot$ and $M = 10^6 M_\odot$.

    Contestants should clarify the concept of *too* uneasy in their solution.

    (Bence Kocsis)

21. Two observers are falling radially towards a black hole, one behind the other. Their initial distance is finite, and their initial speeds are equal. Can they see one another? If they can, in which direction? How does the answer change during the motion? Is there a portion of the trajectory of one observer that can never be seen by the other? And vice versa? (For simplicity, assume that the observers' eyes can see the total solid angle 4π.)

    (Gyula Dávid)

22. Observations establish the fact that in spiral galaxies the circumferential velocity of the stars in circular orbits within the plane of the disk does not depend on their distance from the galactic center. Interestingly, this relation holds even for large distances, where only an insignificant amount of luminous matter is present. In the standard resolution of this contradiction it is assumed that non-luminous (dark) matter is also present in the galaxy, and the gravitational effects of this dark matter take care of the relation at large distances.

    In another possible approach the presence of dark matter is not assumed, but rather Newton's law is modified. Suppose that the gravitational force between two point-like particles is of the form

    $$\mathbf{F} = -G(r)\frac{m_1 m_2}{r^2}\frac{\mathbf{r}}{r},$$

    where $G(r)$ is the distance-dependent gravitational constant. The density distribution of a spiral galaxy can be approximated by

    $$\rho(\mathbf{r}) = \rho_0 e^{-\alpha r}\delta(z),$$

    where r is the distance from the center, and $\alpha \approx 4$ kpc, and z is the coordinate perpendicular to the plane of the galactic disc. How should $G(r)$ be chosen so that the density distribution alone may account for the observations of circumferential velocities?

    Extra question: Can the above form of $G(r)$ be falsified through observations concerning the solar system only?

    (Bence Kocsis and Győző Egri)

23. Study quantum mechanically the motion of a rigid body, of moment of inertia

    $$\mathbf{\Theta} = \begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \end{pmatrix}$$

    ($\Theta_1 > 0$, $\Theta_2 > 0$, $\Theta_3 > 0$), rotating freely about its fixed center of mass.

    a) What is the Hamiltonian of the system?
    b) Find the energy eigenvalues for the spherically symmetrical case $\Theta_1 = \Theta_2 = \Theta_3$. Determine the degree of degeneracy of each level.
    c) Rigid bodies are, however, observed to "rotate" macroscopically, ie their position can be described by time-dependent angle variables. How can this be reconciled with the quantum-mechanical description?
    d) How are the energy levels and the degrees of degeneracy changed, if $\Theta_1 = \Theta_2 \neq \Theta_3$?
    e) What about the general case $\Theta_1 \neq \Theta_2 \neq \Theta_3 \neq \Theta_1$?

    (Bálint Tóth)

24. The Hamiltonian of a quantum-mechanical system is

    $$\mathcal{H} = H_0^{-1}\begin{pmatrix} H_0^2 - \omega^2 & 2H_0\,\omega\cos\alpha & 2H_0\,\omega\sin\alpha \\ 2H_0\,\omega\cos\alpha & \omega^2\cos(2\alpha) - H_0^2 & \omega^2\sin(2\alpha) \\ 2H_0\,\omega\sin\alpha & \omega^2\sin(2\alpha) & -\omega^2\cos(2\alpha) - H_0^2 \end{pmatrix},$$

    ($\hbar$ is set equal to 1), where α is a constant scalar parameter, and H_0 is the well known Hamiltonian of the one-dimensional harmonic oscillator of frequency ω and mass m. Determine (in the simplest possible way) the energy spectrum of the system (and, if possible, the eigenstates as well).

    (József Cserti and Gyula Dávid)

25. The method of "varying effective mass" is often used in the description of inhomogeneously doped semiconductors. The mass of the electron moving inside the solid is then considered as a given function $M(\mathbf{r})$ of the position. The quantum-mechanical treatment runs into difficulties, since the operators $\hat{\mathbf{p}}$ and $M(\hat{\mathbf{r}})$—corresponding to $\mathbf{p}$ and $M(\mathbf{r})$ in the kinetic part of the Hamiltonian $H(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$—satisfy the usual commutation relations and thus do not commute, therefore several different operators can be regarded as the QM counterpart of the classical expression.

Among others, the following expressions have been proposed:

$$a)\quad \hat{\mathbf{p}}\frac{1}{2M(\hat{\mathbf{r}})}\hat{\mathbf{p}}, \qquad b)\quad \frac{1}{4}\left(\hat{\mathbf{p}}^2\frac{1}{M(\hat{\mathbf{r}})} + \frac{1}{M(\hat{\mathbf{r}})}\hat{\mathbf{p}}^2\right), \qquad c)\quad \frac{1}{2}\frac{1}{\sqrt{M(\hat{\mathbf{r}})}}\hat{\mathbf{p}}^2\frac{1}{\sqrt{M(\hat{\mathbf{r}})}}.$$

Examine the one-dimensional case. Show that each of the three expressions above leads to a hermitian energy operator, and that they are related by "gauge transformations". What "compensating transformation" should be performed on the potential $V(x)$? $M(x)$ is assumed to be a continuously differentiable function of the position.

How are the results modified if finite discontinuities may be present in the function $M(x)$ (eg at the interface of two different substances)? What boundary conditions should be prescribed for the stationary wave function solution of the Schrödinger equation in coordinate representation?

(Andor Kormányos)

26. The two-dimensional motion of an electron, including spin-orbit interaction, can be described by the Hamiltonian

$$\hat{H} = \frac{p_x^2 + p_y^2}{2m} + \frac{\alpha}{\hbar}\left(\sigma_x p_y - \sigma_y p_x\right),$$

where α is a constant related to the strength of the spin-orbit interaction, while σ_x and σ_y are the Pauli matrices. In the presence of external magnetic fields, the usual transformation $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$ should be performed on the Hamiltonian, in which e is the electronic charge and $\mathbf{A}$ is the vector potential of the external magnetic field. The Zeeman effect is taken into account by the addition of a further term, $\mu_B \boldsymbol{\sigma}\mathbf{B}$, where μ_B is the Bohr magneton and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector formed by the Pauli matrices.

Determine the electron's energy levels in a uniform magnetic field in the z direction!

(József Cserti)

27. a) Show that for slowly varying rotational motions the component of the vectors along the angular velocity is adiabatically invariant. In other words: consider a twice continuously differentiable function $\boldsymbol{\omega}(t) : [0,1] \to \mathbb{R}^3$, and then choosing an arbitrary number $a \in \mathbb{R}^+$ consider the following differential equation for the vector $\mathbf{r}(t) : [0, 1/a] \to \mathbb{R}^3$:

$$\frac{d}{dt}\mathbf{r}(t) = \boldsymbol{\omega}(at) \times \mathbf{r}(t). \tag{1}$$

Show that, independently of the initial value $\mathbf{r}(0)$,

$$\lim_{a\to 0} \frac{\big(\boldsymbol{\omega}(1/a), \mathbf{r}(1/a)\big)}{|\boldsymbol{\omega}(1/a)|} = \frac{\big(\boldsymbol{\omega}(0), \mathbf{r}(0)\big)}{|\boldsymbol{\omega}(0)|}, \tag{2}$$

where $(\cdot,\cdot)$ stands for the scalar product.

b) Demonstrate that time-dependent perturbation theory is "adiabatically invariant". In other words: consider a twice continuously differentiable function $f(t) : [0,\infty) \to \mathbb{R}$, for which $f(0) = 1$, $\lim_{t\to\infty} f(t) = 0$, and $\lim_{t\to\infty} f'(t) = 0$, and let the integral $\int_0^\infty |f''|$ be convergent. Let H be a known Hamiltonian, and V a perturbation operator. Show that if the eigenvectors of $H + V$ are determined from time-independent first-order perturbation theory, the same results are obtained—up to phase factors—as when an arbitrary number $a \in \mathbb{R}^+$ is chosen and time-dependent first-order perturbation theory is applied to the operator $H + f(at)V$ on the whole interval $t \in [0,\infty)$, and then the $a \to 0$ limit of the obtained result is taken. (In this case the potential is switched off, rather than on.)

What happens in second order? (Take, eg, the function $f(t) = e^{-t}$.) What if the potential is switched off within a finite interval of time, ie for some $b \in \mathbb{R}$, $f(t) = 0$, if $t > b$?

c) How are parts a) and b) related? That is: find (without proof) a general mathematical statement of which a) and b) are special cases. (For simplicity, the appearing vector spaces should considered as finite dimensional.)

(Balázs Pozsgay)

28. Non-commutative quantum mechanics—ie quantum mechnaics with non-commuting coordinate components—has attracted a great deal of attention recently. Examine one of the simplest cases of non-commutative quantum mechanics, a quantum-mechanical model subject to constraints.

Consider the nonrelativistic two-dimensional, planar motion of a charged particle in a uniform and constant magnetic field perpendicular to the plane. The Hamiltonian is

$$H(\mathbf{p},\mathbf{q}) = \frac{1}{2m} \sum_{i,j,k=1}^{2} \left(p_i + \frac{B}{2}\epsilon_{ij} q_j\right)\left(p_i + \frac{B}{2}\epsilon_{ik} q_k\right)$$

The $m \to 0$ limit is studied in a rather intuitive way: to preserve the finiteness of the Hamiltonian, the constraints $C_i = p_i + \frac{B}{2}\epsilon_{ij} q_j$ are imposed. Show that upon quantization, these constraints cannot be imposed on the operator level, ie the original space of states does not possess a (non-trivial) subspace in which the operators C_i yield 0. Instead, proceed as follows:

a) Find a subspace $\mathcal{M}$ and an orthogonal projector $P_{\mathcal{M}}$ (that projects onto this subspace) such that when the operators of the constrained model ($\hat{\mathcal{O}}^c$) are calculated from the constraint-free observables $\hat{\mathcal{O}}$ as $\hat{\mathcal{O}}^c = P_{\mathcal{M}} \hat{\mathcal{O}} P_{\mathcal{M}}$, the operators $\hat{\mathcal{O}}^c$ satisfy the Dirac commutation relations.

b) It will be readily seen that the momentum components $\hat{p}_1^c$, $\hat{p}_2^c$ do not commute any more, and neither do the coordinates. However, the system is symmetric under the transformations of the two-dimensional Euclidean group. Why should this symmetry be spoilt in a suitably defined limit $m \to 0$? What representation of the translations is derived from the usual representation in the original, constraint-free model? Isn't it possible to get rid of the noncommutativity of the momentum components by making a transition to an equivalent *ray* representation?

c) When quantizing, an operator is associated with the classical monomial $p_1^n p_2^m$, which is a polynomial of $\hat{p}_1^c$ and $\hat{p}_2^c$. What ordering rule is found for the momentum component operators?

d) Let f and g be functions of the coordinates alone. Define a (not necessarily commutative) $*$ product that satisfies the equality

$$P_{\mathcal{M}}\,(f * g)(\hat{q}_1,\hat{q}_2)\,P_{\mathcal{M}} = P_{\mathcal{M}}\,f(\hat{q}_1,\hat{q}_2)\,P_{\mathcal{M}}\,g(\hat{q}_1,\hat{q}_2)\,P_{\mathcal{M}}.$$

What is the $*$ product of the monomials $q_1^k q_2^l$, $q_1^m q_2^n$?

(Szilárd Farkas and Márton Kormos)

29. Consider a tiny drop of superfluid helium (^{4}He) of zero temperature. As it is well known, sound waves propagating within the droplet can be described by a suitable effective quantum field theory, and the quanta of the sound waves are called phonons. According to the uncertainty principle, a zero-point vibration is associated to each phonon mode. What energy density is created in the helium droplet by the zero-point vibrations? What pressure is due to this energy density? Why does not this pressure make the droplet explode or collapse?

(Hint: the maximum possible energy of the phonons is the Debye energy. Perform the arising calculations for an interacting *Bose gas*, and then try to deduce implications on the quantum fluid.)

(Győző Egri)

30. Two small ion clouds collide. Each contains a very large number of the same ion. Electromagnetic interactions among the particles are ignored; the phenomenon is governed by the very short-range strong interaction. The probability that at least one intercloud collision (ie, a collision between one ion in one cloud and another ion in the other cloud) occurs is known, and is denoted by p ($0 < p < 1$). What is the probability that at least *two* intercloud collisions occur?

(Gábor Veres)

31. Suppose that the distribution of various characteristics of terrestrial index forms is scale-independent. For example, suppose that the distribution of the area of the islands, the height of the mountains, the length of the rivers or the area of their watershed each shows a scale-independent distribution with some exponent. Try to show about these and other similar distributions that they are indeed scale-independent. Construct models that relate the exponent of various distributions. Verify these relations against measured data (taken from maps or databases). Does the topology of the earth's surface enter some of the relations among the exponents?

(Merse Előd Gáspár)

32. That's it, I'm fed up with the Ortvay contest. Too many problems, too little time; the pressure drives me truly crazy. I took a ten-minute break, and ended up with back-of-the-toilet-paper calculations… But enough is enough, I take the roll, lean out of the window on the top floor of my skyscraper, and start a new experiment.

I press the end of the paper strip against the wall, and observe how it rolls down along the wall. A part stays in my hand, and there you go again, calculations start…

How does the height of the center of the paper roll change with time? When and where does the paper break (if it does)?

(Gyula Dávid and József Cserti)

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