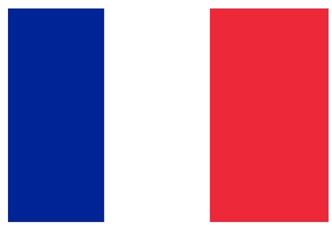
THE 48th—20th INTERNATIONAL—RUDOLF ORTVAY PROBLEM SOLVING CONTEST IN PHYSICS

27 October— 6 November 2017

1. After the french revolution, the results of sciences have been widely applied and propagated. Such results were of consideration even during the design of the new french banner, the tricolore. Lafayette created two different style of the flag, which only differed in the proportion of the areas for the three color.

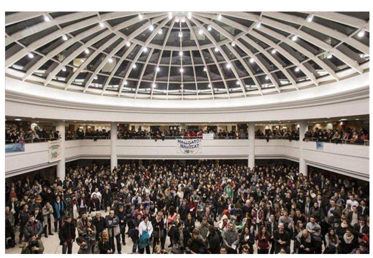
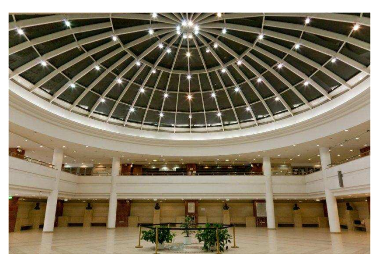
   One variant had equal areas of blue-white-red, while the other had 30:33:37 proportions. The official banner of the country has been one or the other at different eras, altough the navy mostly used the latter version.

   Give qualitative explanation for the usefulness of unequal division. Investigate the question qualitatively as well.

   (Ákos Gombkötő)

2. The North Building in the Lágymányos Campus of Eötvös University features a nearly hemispherical hall, which is quite characteristic from its echoing.

   If one stands in the middle, the strength of the echo from our own speech reflected from the ceiling is surprisingly strong. If we clap the hands, there is a periodically repeating echo which gets gradually quiet. One would think naïvely that the repetition period is the same as the time needed for the sound to reach from the floor to the ceiling and back. However, based on the height of the hall $h \approx 10.8$ m in the center and assuming sound velocity of 344.7 m/s, we get around 0.063 s, which is evidently too frequent by practical experience. Even without measurement, one finds that the number of repetitions is around 4–5 per second. Why is this so, and how much the repetition time should be? For the calculation, it is relevant to know that the radius of the hemisphere is $R \approx 15.2$ m, that is, the spherical surface, completed down to the level of the floor, would actually not give a complete hemisphere. Assume that the floor is a horizontal plane. *Suggestion:* let us first study the sound waves which pass close to the vertical symmetry axis!

   (Zoltán Kaufmann)

3. I was heading towards a bus terminal with my son during a steady rain. When we walked on a long ramp, we have observed that the water was not flowing continuously, but there were waves at roughly equal distances, rolling down pretty rapidly on the surface.

   The water supply was continuous on the elevated end, so the emergence of the waves was not apparent, but the resulting pattern was spectacularly regular.

   Why do these waves appear? What is their velocity? What is the time interval between waves?

   (Based on Éva Tichy-Rács's observation Ádám Tichy-Rács)

4. Statistics very often uses coin flipping as an illustration for stochastic process. Without doubt, a real coin flipping, accounting for all details is a truly complicated phenomena, but with certain simplifications, mechanical flipping with well defined parameters in a vacuum chamber, the results can be predicted to a certain extent.

   Investigate the dynamics of a vertically thrown, thin coin with small radius ($r \gg z$) from the initial height of the floor.
   Initially the coin rests horizontally, the side with the head on the top. The coin does not spring back from the floor once it touches it, drag forces are negligible, the radius r of the coin is negligibly small compared to the characteristic length of the trajectory of the center of mass.

   Examine whenever will the result be head, and tails, as a function of the parameters of the flipping (velocity, angular velocity). Formulate the qualitative criterion of the stochasticity.

   Write an upper bound for the quantity $|P_{\text{head}} - 1/2|$.
   Approximately what would be this value for a real coin-flipping machine?

   (Ákos Gombkötő)

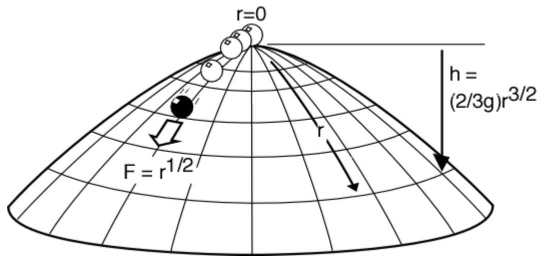
5. As Christmas approaches, here and there the shiny baubles (spherical reflecting ball ornaments) appear. Where a few such balls are close to each other, there can be multiple reflections, and distant lamps appear as multiple bright spots with their multiply reflected lights. Let us study for simplicity the case when two spheres of radius r are placed at distance d, and illuminated from a direction perpendicular to the line connecting their centers, and also viewed from that direction. Assume that both the illuminating lamp, and also the observer is very far, that is, the light from the lamp can be considered to be parallel rays. Where do the reflected bright spots appear on the ball, specifically, which rays will be reflected back to the direction of the illumination after $n = 1, 2, 3$ etc. reflections? Are there such a ray which bounces infinitely many times between the spheres? Asymptotically, which is the sequence of the points in the limiting case $n \to \infty$, and can one calculate analitically any parameter of the sequence? If numerical calculation is needed, let us study the case $d = 2r$.

   (Zoltán Kaufmann)

6. Give such $V(r)$ central potentials, in which mass points, starting from a given point at a radius R, with the same speed but in different directions, will run along the following peculiar orbits.
   a) All orbits end up in the same point, diametrically opposite to the starting point.
   b) Once all orbits reach the radius R again, they continue as parallel lines and thus leave towards infinity. (Two different potentials should be given for the respective cases, together with reasonings, and in each case the energy and the directions of the initial velocity whereby such orbits can be realized should also be determined.)

   (Géza Györgyi and Máté Vigh)

7. Examine the movement of a body on the following, Norton-dome.



Here the parameter r is the length of arc measured from the top. Not too far from the top the equation of motion for a point mass is $\ddot{r} = \sqrt{r}$. Derive this equation of motion, then write down the general solution for the case when in $t = 0$ the body rests on the top.

Philosophers have written numerous articles about this system. Explain what the reason for this could be. Show in at least two different ways how this system as a model fails as a mapping of the real world.

Show that an equivalent system can be solved, without any particular curiosities arising.

Study the quantum version of the Norton-dome, where the wavefunction in bounded to the surface. What will the energy eigenstate be, which have exactly the energy that would be neccessery for a classical particle to rest on the top? The angular momentum of this state can be assumed to be zero.

(Ákos Gombkötő)

8. Determine the harmonic frequencies of a linear chain, which is free on both ends, and is built up from N equal masses connected with equally strong springs.

(Géza Tichy)

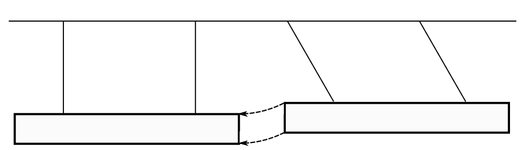
9. A sphere of radius R is built up from a homogeneous 'Northern' and 'Southern' hemispheres, but the hemispheres differ in mass, M_1 and $M_2 \neq M_1$. a) Determine the work needed to bring a pointlike mass m from the North pole to the South pole. b) Calculate the work between two arbitrary points on the surface of the sphere. Consider only the gravitational field of the sphere.

(József Cserti)

10. The 'gomboc' ('globule') is a homogeneous convex object which features a single stable (and also single unstable) equilibrium when placed on a flat surface. Its specific shape was invented by Hungarian mathematicians, however, there are infinitely many possibilities. Can we give an upper and lower bound for the frequencies of the small amplitude oscillations on the flat horizontal surface? Assume that the body does not slip, but only slightly rolls.

(Ákos Gombkötő)

11. There is a method to determine the Young modulus of very hard solids. Two uniform rods are hanged horizontally as two pendulums. One of them is displaced and released. The rods collide, and after the collision the other is displaced in the opposite direction. The time of the contact during the collision is measured by an electric watch. Determine the Young modulus of the rods from the collision contact time T, the cross section area, the mass and the length L of rods.



(Géza Tichy)

12. A planet is orbiting around a fixed star, on an ellipse with major axis a and eccentricity ε. Let us translate from the inertial system fixed to the star to an other inertial system, which moves from the center of the ellipse towards the star at a constant velocity $v = c\,\varepsilon$, where c is the speed of light. Which is the orbit of the planet seen from this new coordinate system? Let us determine, and plot the coordinate representing the position as a function of the new system time (or the other way around).

    Suggestion: for the parametrization of the orbit, let us use the quantity 'eccentric anomaly', which is pretty liked by astronomers.

    (Gyula Dávid)

13. In the classical mechanics, the conservative systems are typically considered reversible. On the other hand we know, that the ideal gas, which can be modeled using mechanics without any kind of dissipation, behaves seemingly irreversibly.

    Let us consider a similar system.
    Let us have a chosen harmonic oscillator, which is linearly coupled to N different harmonic oscillators. For arbitrarily large N, the eigenfrequencies are real.
    The displacement of the chosen harmonic oscillator is X, while the vibrational modes of the environment can be described by x_i, where the index i denotes the modes for countable N. For uncountable N, x_i has displacement-density meaning.

    The case of the continuum N modes is special, because for weak coupling, simple analytical solution is possible. The Lagrangian for the system is

    $$L = \frac{m_0}{2}(\dot{X}^2 - \omega_0^2 X^2) + \int_0^\infty \frac{\rho(\omega)}{2}(\dot{x_i}^2 - \omega^2 x_i^2)d\omega - X\int_0^\infty C_i x_i d\omega,$$

    where the environment is made of oscillators with a wide range of frequencies, $\rho(\omega)$ is the density of states weighted by mass, $C(\omega) \ll \rho(\omega)\omega^2$ is the density of state weighted by the oscillator-environment coupling strength. What can we say about the eigenfrequencies?

    Calculate the time-evolution of the X, and write down the effective differential equation that governs it.
    Describe the system with the definitions in the literature. Is this system conservative? Is this system dissipative?

    (Ákos Gombkötő)

14. Ernst Chladny died 190 years ago (https://en.wikipedia.org/wiki/Ernst_Chladni), who could create beautiful patterns on a sand covered plate with a violin bow. Is it possible to create any given pattern on a vibrating plate? If yes, let us try to make the name CHLADNI appear in a simulation (or even possibly in an experimental arrangement). The plate can be assumed to be an ideal membrane, either with fixed or free boundary conditions. If we do not only look for the nodal lines, but observe the value of the amplitude, can one create 'grayscale' images, such as Chladni's portrait? If one can not create any arbitrary image, let us explain the limitations.

    (István Csabai)

15. We place a thin capillary tube of radius r vertically on the surface of a liquid with very low viscosity, in a way that the surface just contacts the tube.

    For the viscosity, it stands that $\eta \ll g\rho^{3/2}r^{5/2}\gamma^{-1/2}$, where g is the gravitational acceleration, ρ is the density of the liquid, r is the inner radius of the tube, and γ is the surface tension.
    We can assume that the liquid is wetting, and can consider the height of the meniscus to be equivalent with the height of the pressure-column.

    Describe the one-dimensional dynamics of the height of the meniscus. What is the maximal ratio between the maximal height and the equilibrum height?

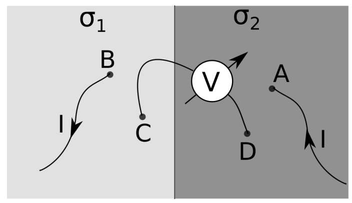
    (Ákos Gombkötő)

16. In the corners of a regular tetrahedron, pointlike equal charges are fixed. In the center of the tetrahedron, there is a small, finite mass charged body, which can move along a straight line without friction.

    If the central body is displaced by a distance of A and released, the period of the oscillation will be $T(A)$. How much is the period time $T(2A)$ for twice larger displacement?

(Péter Gnädig)

17. An infinite metallic sheet of negligible thickness is made from two half-planes with electrical conductivity σ_1 and σ_2, which are perfectly fitted. Find the resistance of any four-point measurement $R_{AB,CD} = \frac{V_{CD}}{I_{AB}}$, if the current I_{AB} is injected into contact A and taken out of contact B, furthermore, the voltage V_{CD} is measured between contacts C and D.



(József Cserti and Gábor Széchenyi)

18. In the left ventricle of the human heart the blood pressure varies approximately periodically. However, though the movement of the heart muscles is periodic, the blood supply needs to be practically continuous. Let us model the blood stream assuming a series heart – aorta – arteries (blood vessel) system! One can consider the aorta to be a flexible chamber, whereas the the blood vessels present a constant R hydrodinamic resistance. Calculate the time dependence of the volumetric current output at the end of the blood vessels, if the current from the heart depends on time as $I(t) = I_0 \sin^2 \omega t$! (*Suggestion:* let us be critical with the relevant literature!)

(Ákos Gombkötő)

19. We have learnt at school, that if it's hot, then the ears of a chocolate bunny will melt, since it sticks out of the body. Let us verify this using a numerical modeling in a simplified case! Assume a cylinder of radius r and length h, which is homogeneous water ice at temperature $-10°$C. The cylinder floats in air at the International Space Station, and does not touch anything besides the surrounding air. There is no heat radiation: only the air heats it on the surface, which is at temperature $+20°$, can be assumed to be infinitely large volume, and does not move. Let us determine at which values of r and h will the ends of the cylinder turn to a „spike", and when we get a „sharp-edged" disc – provided that these forms appear at all. For the study, let us first define sufficiently precisely these categories. Qualitatively, which shapes can be formed during the melting?

(Gábor Veres)

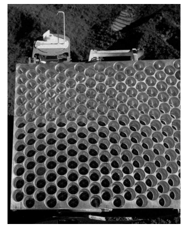
20. How much higher do the illumination increase on a white surface (detected by eye), if the temperature of the tungsten wire of an incandescent lamp bulb rises from 2700 K to 3000 K? How much does it increase the efficiency of the lamp?

(Imre Sánta)

21. Study the imaging properties of a flat material of thickness d which features a negative refractive index. Assuming realistic cases, estimate the magnitude of the chromatic aberration. How can we reduce this chromatic aberration without using correction lenses?

(Ákos Gombkötő)

22. A number of Soviet and USA Moon-mission carried retroreflector arrays onto surface of Moon. The greatest array from those (0.68 m^2) left by the Apollo 15 mission, placed near perpendicularly to direction of the Earth. It consists of 300 individual corner cube prisms of diameter 38 mm. On the Earth, we expand the beam of a Q-switched Ruby laser (energy 1 J, TEM_{00} mode, 10 ns FWHM pulse width) by guiding the beam to a telescope of diameter 3 m. We aim the beam centre to the retroreflector array on the Moon as accurately as possible. What will be the size of the reflected beam on the surface of Earth? How many photons will arrive to the focal point of the same telescope after about 2.5 s from the laser light and what is the temporal distribution of them?



(Imre Sánta)

23. In order to determine the charge to mass ratio of an electron, we use a small sized cathode ray tube. The electrons are accelerated before the deflection electrodes, the average potential of these latter electrodes is equal to the last accelerating anode potential. The accelerating voltage is $U_a = 1.15$ kV. The electrons exit from the accelerating region through a very small hole, but not yet focused: they would present a one centimeter diameter spot on the screen. Focusing is achieved with a coil, having an axis parallel with the cathode ray tube. The length of the deflection plates is $x_1 = 23.45$ mm, the distance between the last anode and the screen is $x_2 = 234$ mm. In fact textbooks call the deflection plates as „capacitor”, and on drawings they appear as parallel planes, however, in principle they can not be parallel. And they are not capacitor as well. Other relevant construction parameters of the coil are: length is $L = 450$ mm, diameter is $D = 49.5$ mm, and the thickness of the wire (including insulation) is $d = 0.35$ mm. The turns of the coil are wound tightly next to each other. Let us assume that there are no ferromagnetic materials inside the cathode ray tube. The charge to mass ratio can only be determined if we can focus the electrons on the screen, because only then we can infer the trajectories of the initially slightly divergent electrons. During the measurement, we increase the current from $I = 0$ continuously. During the design of the system, we can assume that the current to mass ratio of the electrons is known. Calculate the velocity of the electrons before impacting on the screen! Determine the minimal value of the magnetic field $B_{\min}$ and the corresponding current $I_{\min}$, when we get a focused electron beam on the screen for the first time!
Hint: Consider an experience with a colleague, when we intended to have a beer together. We stepped out of the school building together at the same time, precisely at 17:01:02. I was walking straight to the pub, whereas my colleague had some appointment so we took different routes. Still, we arrived at the same time, precisely at 17:18:19. At this point we realized that the problem above is considerably simpler than we thought.

(István Bartos-Elekes)

24. Consider two given radioactive atoms. The probability that one of them will decay in time interval $[t; t + dt]$ and the other one will decay in time interval $[s; s + ds]$ is given by

$$N \exp\left(-as - bt - c\sqrt{st}\right) ds dt, \qquad a, b > 0, \quad c \in \mathbb{R}.$$

Determine the normalization factor N of the probability density function in analytical form if it is only defined in the first quadrant. Plot the probability in time interval $T \in [0, 100]$ that both atoms decay within time T, choosing appropriate parameters. Determine numerically the approximate probability that one of the atoms decays earlier than the other one, choosing appropriate parameters. Calculate the expected value and standard deviation of decay time for both atoms in analytical form.

(Mihály Csirik, Gábor Homa, László Lisztes)

25. In classical or relativistic electrodynamics, the coupling between point-like particles and the electromagnetic field is described by a scalar coupling constant, the electric charge. Let us study the situation when the coupling is realized by an anti-symmetric four-tensor μ_{kl}, containing the actual coupling constants. Elaborate the complete action integral of the particle and the electromagnetic field (in vacuum), including the new terms with the coupling tensor. (If we consider the components of the coupling tensor to be new dynamical degrees of freedom, we can include in the action integral the term describing the dynamics of this quantity, taking the covariant Lagrangian form $\dot{\mu}_{kl}\,\dot{\mu}^{kl}/2$, where the dot represents the derivative with respect to proper time of the particle).
Derive the equation of motion of the particle, and the new forms of the Maxwell-equations as well as the equations for the dynamics of the tensor μ_{kl}. Rewrite the equations also in the three dimensional notation. (*Suggestion*: the antisymmetric four-tensor μ_{kl} can be described by two three-dimensional vectors, $\mathbf{p}$ and $\mathbf{m}$.). Try to find conservation laws based on the symmetries of the system.

(Gyula Dávid)

26. Assume that we bear the 'cosmological principle' in mind, and we are looking for a velocity field that describes the expansion of the matter that fills the whole space.

Non-relativistically, the (generalized, elliptical) self-similar Hubble flow profile ($v_x(\mathbf{r},t){=}H_x(t)r_x$, $v_y(\mathbf{r},t){=}H_y(t)r_y$, $v_z(\mathbf{r},t){=}H_z(t)r_z$, where $H_k(t)$ are arbitrary functions of time t) has the convenient property that it satisfies the cosmological principle, inasmuch as for *every* time t and space vector $\mathbf{r}$, if an observer is at the coordinate $\mathbf{r}$ moving with the corresponding velocity $\mathbf{v}(\mathbf{r},t)$, then (according to the Galilei transformations) he sees an expansion with a velocity field around him that is identical to the one seen by the observer at rest at the origin ($\mathbf{r}$=0.)

The task is to find the most general $\mathbf{v}(\mathbf{r},t)$ velocity field that generalizes this property to the case of special relativistic kinematics (i.e. Lorentz transformations). In this case, we have some conceptual freedom in the choice of the time at the origin for what we require that the velocity field seen from there has identical appearance as the one seen by the moving observer. We can try three different possibilities: the compared timestamps should be simultaneous 1) in the rest frame, 2) in the frame of the moving observer, or 3) in a frame from which the origin and the moving observer has the same (but opposite) velocities. (We can restrict ourselves to the spherically symmetric case: $\mathbf{v}(\mathbf{r},t) = \mathbf{r}\cdot f(|\mathbf{r}|,t)$, and we can assume that at the origin $\mathbf{v}$=0.)

(Márton Nagy)

27. Satellites are orbiting Earth at various radii on circular orbits. For simplicity, let us consider only those which orbit in the plane of the Equator. Precise on-board timing instruments allow us to measure the relativistic time dilatation effects, as well as gravitational redshift biases according to the rules of general relativity. Are there among the satellites, measuring their proper time, which are exactly synchronous with the observers on the Earth surface (on the Equator)? If so, how high is the orbiting altitude? (Express this as multiple of the Earth radius, as well as in units of km!) How much is the difference quantitatively if one disregards, or takes into account, the rotation of Earth? For the calculations, consider Earth to be perfectly spherical.

(Gyula Dávid)

28. A nonrelativistic point-like charge is moving in a time independent magnetic field that is invariant under translations along the z-axis. The canonically conjugate momentum of the position operator $(\hat{x},\hat{y},\hat{z}) \equiv \vec{\hat{r}} = \vec{r}\cdot$ of the particle is chosen to be represented by $\vec{\hat{p}} = -i\vec{\nabla} + q\vec{A}$, where $\vec{A}$ is the vector potential, and q is the charge of the particle ($\hbar = c = 1$). Under which conditions the above operators represent the canonical commutation relations? Show that as opposed to expectations even if these conditions are satisfied, there exists a simple, nontrivial system in which this representation is not of Schrödinger type, i.e. such a unitary operator $\hat{U}$ does not exist, for which $\hat{U}^\dagger\vec{\hat{p}}\hat{U} = -i\vec{\nabla}$!

(Gergely Fejős)

29. Consider the spin of a localized electron in a homogeneous magnetic field, $\vec{B} = (0, 0, B_0)$. The Hamiltonian describing the spin is $\hat{H} = -\frac{1}{2} g \mu_B \vec{B} \cdot \hat{\vec{\sigma}}$, where $g = 2$, μ_B is the Bohr magneton, and $\hat{\vec{\sigma}}$ is the vector of the three Pauli matrices representing the spin. Initially, the spin is in the ground state, that is, its polarization vector $\vec{p} = \langle \psi | \hat{\vec{\sigma}} | \psi \rangle$ points along the z axis. Let us try to rotate the spin, via rotating the magnetic field vector with a constant angular velocity in the xz plane: $\vec{B}(t) = B_0 \left(\sin(2\pi f t), 0, \cos(2\pi f t)\right)$. After rotating the magnetic field, that is, at time $t = 1/f$, we measure the z component of the spin. What is the probability of measuring $+1$, $P_+(B_0, f) =$? Can you conjecture the result in the $f \to 0$ and $f \to \infty$ limits, without actually calculating it? Plot the function $P_+(f)$ in the $f \in [0; 100\,\text{GHz}]$ interval in the case of $B_0 = 1$ Tesla. Plot the trajectory of the polarization vector in the cases $f \in \{0.1; 1.0; 10.0; 100.0\}$ GHz.

(András Pályi)

30. Consider a quantum-mechanical system composed of two $s = 1/2$ spin objects. In the time interval $t < 0$ its Hamiltonian $\hat{H}_0$ is spin-independent; however, at $t = 0$ due to some abrupt external change a new term appears in addition to which can be written in terms of spin operators in the form

$$\hat{H}' = \frac{4\,\Delta}{\hbar^2} (\mathbf{S}_1\, \mathbf{S}_2).$$

This new term then persists for all $t > 0$ in identical form.

Let us assume that for $t \leq 0$ the system is in the state

$$|+-\rangle = \frac{|1,\, 0\rangle + |0,\, 0\rangle}{\sqrt{2}}.$$

Given the perturbation above, determine to the lowest order of approximation the probability that the time evolution of the system will terminate in each one of the final states $|++\rangle$, $|-+\rangle$, or a $|--\rangle$.

Suppose the result obtained is inconclusive (*Hint*: it may well be...). Perform a suitable more advanced calculation, then try to explain the reasons for the inadequacy of the naïve approximation in the light of the more accurate results so obtained.

(Péter Magyar)

31. Consider the following wave function of an $S = 2$ spin:

$$|\psi\rangle = \frac{|2\rangle\ - |\bar{2}\rangle}{\sqrt{2}},$$

where the $|2\rangle$ and $|\bar{2}\rangle$ are the eigenstates of the S^z spin operator with eigenvalues $+2$ and -2, respectively.

a) What are the expectation values of the spin operators $\mathbf{S} = (S^x, S^y, S^z)$? Show that $|\psi\rangle$ is time reversal invariant.
b) What SO(3) rotations transform the state $|\psi\rangle$ into itself (up to a phase)? Show that these rotations form a group isomorphic to the dihedral group D_4. What are the Berry phases the wave function picks up during these rotations ? To get an insight, it might be helpful to consider directions $\hat{\mathbf{n}}$ for which $\langle \hat{\mathbf{n}} | \psi \rangle = 0$, where $|\hat{\mathbf{n}}\rangle$ is a spin coherent state defined by $(\hat{\mathbf{n}} \cdot \mathbf{S})|\hat{\mathbf{n}}\rangle = 2|\hat{\mathbf{n}}\rangle$ and $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ is a unit vector.
c) What is the space of the wave functions obtained by all SO(3) rotations? How is the Berry phase related to the the first homotopy group of this space?

(Karlo Penc)

32. Let us consider some quantum-mechanical system whose Hamiltonian has two eigenvalues ($E_+ = \hbar\omega$ and $E_- = -\hbar\omega$) with corresponding (normalised) eigenstates $|+\rangle$ and $|-\rangle$. Further assume that at $t = 0$ the system is prepared in the state

$$|\psi_0\rangle = \frac{1}{2}|+\rangle + \sqrt{\frac{3}{4}}|-\rangle.$$

Consider also some observable $\hat{A}$ of the above system with two eigenvalues a and b and corresponding eigenstates

$$|a\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \qquad |b\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}.$$

Investigate the outcome of the following two scenarios.
a) The system evolves from $t = 0$ to some arbitrary $t = 2T$. Then a physicist measures the value of $\hat{A}$. What is the probability of finding the value a in this measurement?
b) In the subsequent control measurement, the system evolves from $t = 0$ as before until the time $t = T$ is reached. However, at that instant the physicist performing the measurement receives an unexpected mobile phone call, and distracted by it accidentally hits the Enter key measuring $\hat{A}$. Trouble – no time no way to repeat the experiment. Without any better way out, our physicist decides NOT to read the value so obtained, hoping that without intervention the system did not collapse so nothing happened. Then s/he lets the system evolve again until $t = 2T$, repeats the measurement of $\hat{A}$, and somewhat relieved proceeds to calculating the probability of obtaining the value a.

Compare the two cases and comment: Can our imaginary colleague really relax or not?

(Péter Magyar)

33. A well-known problem in quantum mechanics is that neither momentum eigenstates, nor position eigenstates are elements of Hilbert space, hence using these as basis is non-trivial. While the energy eigenstates do give a complete basis, but unlike the former, these are not 'a priori given' and calculating them is often not simple.

In certain aspects it might be more adequate to use a 'known' basis of the Hilbert space. One such example is the coherent-state basis.

Choose a complete, minimal basis made of coherent states.
Analyse the harmonic oscillator in such a basis. Expand the number-eigenstates, then check the results numerically as well.

(Ákos Gombkötő)

34. A quantum particle of three-component wave function can be described by the following Hamiltonian:

$$\hat{H} = \frac{\mathbf{p}^2}{2m}\hat{I} + \frac{\hbar\omega}{\mathbf{p}^2}\,\mathbf{p} \circ \mathbf{p},$$

where m and ω are positive parameters, $\hat{I}$ is the unit matrix of size 3×3, $\mathbf{p}$ is the momentum operator of the particle, and the sign $\circ$ denotes the dyadic product.

Calculate the position operator $\mathbf{x}(t)$ of the particle as function of time in Heisenberg picture. Determine the time dependent expected value of the position operator in the case of a given (e.g. Gauss-like) initial wave-packet. Give a physical interpretation of the results.

(Gyula Dávid)

35. The ‘Suricate Deep Space Base’ is built from N space stations which are placed on the corners a regular polygon (N-gon), at fixed positions relative to each other. The researchers working on the space stations – just like the suricates in the wilds – continuously watch their surroundings, to gather as much data as possible. Besides others, they observe the spectrum of a hydrogen atom, positioned in the middle of the polygon at rest. One day, a ‘cosmic storm’ charges all the space stations to the same electric charge Q – at least, this is the conclusion of the researchers, after experiencing that the spectral lines of the observed central hydrogen atom shift and new lines appear as if known lines would split. Let us follow the thinking of the scientists! Determine the splitting of the p, d, f orbits of the hydrogen atom for an $N = 42$ polygon. Can one determine the value of Q? How do the positions of the spectral lines depend on the value of N? Why is the case $N = 4$ special? Let us study the case when $N \to \infty$ while the NQ product is constant. For the calculations, we may assume that the hydrogen atom is sufficiently far from the space stations.

    Let us calculate for a 100 km diameter space base the value of Q which causes an effect comparable to the hyperfine splitting or the Lamb-shift! How do the spectral lines shift if the space stations, charged to Q, are slowly rotating together in the plane of the polygon, around the center?

    (József Cserti)

36. Extreme short light pulses can be generated using the so called high harmonic generation (HHG). For a long time, the result of this process has been modeled mostly classically/semiclassically. Many important aspects of this process can only be hoped to be described appropriately through an appropriate quantum model.
    Let us take the simplest model for which we can realistically expect the qualitatively correct description of the HHG. Take a two-level system, which is linearly coupled to infinite amount of electromagnetic modes, and which interacts with a strong, classical, short laser pulse.
    You can refer to the following article for the exact form of the model, as well as certain acceptable considerations: (https://arxiv.org/abs/1605.01087)
    Give an analytical approximation for the time evolution of the expectation-value of the photon-numbers. You can write solutions for either general laser pulse, or for special cases. The initial conditions may be chosen by you.

    (Ákos Gombkötő)

37. Consider the following hierarchical system of four point masses: two stellar-mass ($m_1 = m_2 = 10M_\odot$) and two supermassive black holes ($m_3 = m_4 = 10^{10}M_\odot$). Let m_1 and m_2 move on a circular orbit with radius $a_{\rm in}$ around its common center of mass, and its center of mass orbits the supermassive binary system with radius $a_{\rm out}$, where $a_{\rm out} \gg a_{\rm in}$. Let us refer to the two orbits respectively as inner and outer orbits. The separation of the supermassive black hole binary R satisfies $R \ll a_{\rm out}$.
    a) Make an order-of-magnitude estimate on what is the separation ratio of the inner and outer orbits $a_{\rm in}/a_{\rm out}$ such that this configuration is stable against the tidal field of the supermassive binary?
    b) Let us assume that the supermassive black hole binary suddenly collides and releases $\varepsilon = 10\%$ of its mass in gravitational waves. How do the semimajor axis and eccentricity of the inner and outer binary change?
    c) Can the originally stable inner or outer binary be disrupted due to the effects of gravitational waves for any choice of ε and $a_{\rm in}/a_{\rm out}$? If so, for what parameters does either the inner or the outer binary get disrupted?

    Let us use non-relativistic Newtonian classical physics for the estimate assuming that the orbital distances $a_{\rm in}$ and $a_{\rm out}$ are much larger than the black hole horizon radii.

    (Bence Kocsis)

38. LIGO and VIRGO have recently detected gravitational waves from merging black hole and neutron star binaries opening a new window on the Universe. For all sources to date, the gravitational wave frequency and the amplitude increased in time until the system merged. For circular orbits, the dimensionless amplitude of the gravitational wave strain is approximately $h = (32/5)^{1/2} G^2 M\mu/(c^4 rD)$, where $M = m_1 + m_2$ and $\mu = m_1 m_2/M$ are the total and reduced masses and r and D are the orbital radius and the distance to the source, respectively, and the gravitational wave frequency is twice the orbital frequency. In the leading order approximation, the merging binary follows a Keplerian orbit whose semimajor axis and eccentricity are slowly changing due to gravitational wave emission as follows:

$$\frac{da}{dt} = -\frac{64}{5}\frac{G^2 \mu M^2}{c^5 a^3} F(e),$$
$$\frac{de}{dt} = -\frac{304}{15} e \frac{G^2 \mu M^2}{c^5 a^4} H(e),$$

where $F(e) = [1 + (73/24)e^2 + (37/96)e^4]/(1-e^2)^{7/2}$ and $H(e) = [1 + (121/304)e^2]/(1-e^2)^{5/2}$. Thus, for isolated binaries the Keplerian orbital frequency and the gravitational wave frequency increase.

LIGO and VIRGO are also looking for unexpected astrophysical sources of gravitational waves. Can we think of configurations of point masses, where in contrast to the previously known sources, the frequency of a gravitational wave source *decreases* in time? In such a configuration, what is the time-dependence of the orbital elements?

Let us make simple leading order estimates using Newtonian non-relativistic physics where possible.

(Bence Kocsis)

39. Mauritz Cornelis Escher died 45 years ago, who created pictures respected by many scientists. Study the behaviour of an Ising-model, arranged according to his picture „Circle limit IV" (http://www.mcescher.com/gallery/recognition-success/circle-limit-iv/) with ferromagnetic and anti-ferromagnetic coupling, and compare with the usual two dimensional square grid model. Is there any phase transition, and if so, at which temperature?

Consider and elaborate the finite size effects as well.

(István Csabai)

40. Captain Pirx orbits in the vicinity of a vibrarium cloud with his reconnaissance ship, in a low density gas where decelerating effect due to friction or drag is negligible. As we well know, the vibrarium is vitally important for the development of space-warp-propulsion allowing travel faster than light. As a matter of fact there is only very small amount available in the Known Space, no wonder that a heated competition exists among intelligent space-faring civilizations to get hold of the vibrarium sources.

And now here it is, a new reserve, as much as a planet (approximately the same total mass as Earth), clean vibrarium gas cloud. First measurements reveal that it is spherically symmetric, with a density exponentially decreasing from the center in the radial direction. In this gaseous form, vibrarium is innocent, does not have any space-time curving or biasing effect (unlike the case when used in the space-warp-propulsion engine). Therefore, Pirx starts to orbit the outer part of the apparently ordinary gas cloud, taking good care that none of the concurrent (or friendly – so difficult to tell apart friend or foe!) civilizations can get hold of the reserve.

Indeed, his worries were not unjustified. Taking just a half turn around the center of the cloud, he sees that on the other side of the cloud, exactly opposite to him, an other ship arrives from infinity, breaks, and starts to orbit the cloud at a circular orbit having exactly the same radius as of Pirx. He turs all the on-board instruments towards the other ship (only passive instruments, of course, as he does not want to reveal his presence). Fortunately, the vibrarium gas is rather transparent to visible light, and the relevant Smith – Kovács – Kuznetsov theory states that the refractive index is dependent only on the density. It is surprising that the strange spaceship, appearing on Pirx's display, is pretty much similar that his own, even the ensign of the United Planets Federation is recognizable on the tail. Pirx remembers the lesson from the Space Academy – all is suspicious what is not suspicious. Sure, the enemy disguises their spy ships as Federation ones... Let's take a closer look! Pirx brings his vessel to the deeper regions of the cloud, assuming that he gets nearer by faster orbiting. Alas, after half an orbit, he observes that the foreign ship dives deeper into the cloud as well, and takes an orbit of the same radius as Pirx, exactly at the opposite position. Wants me no closer – Pirx thinks – but I can not let it be so! – He initiates then a new maneuver.

You can guess: the sequence of events above repeats an other forty one times. Every time that Pirx moves to a new orbit at a different radius, the strange ship follows after half a turn, and takes an orbit at the same radius in the opposite side.

After the forty second unsuccessful attempt (this is stipulated by the Service Code and the Hitchhiker's Guide), Pirx gives up this futile spacecat-and-spacemouse game, and leaves the vibrarium cloud. Half a turn later the foreign ship follows suit. Pirx reports at the capital ship – the commander considers him a lunatic, as the fleet's very sensitive instruments did not reveal any unidentified space operations within lightyears.

As for now, only time will tell which civilization can take advantage of the vibrarium reserve, just as for the fate of Pirx's career. From the scientific aspect, we are concerned about one question: precisely which is the formula describing the dependence of the refraction index of the vibrarium gas on its density?

(Gyula Dávid)

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