THE 30th
—2nd INTERNATIONAL—
RUDOLF ORTVAY
PROBLEM SOLVING CONTEST IN PHYSICS
1999

The Physics Students' Association of Eötvös University, Budapest, the Roland Eötvös Physical Society and the Hungarian Association of Physics Students proudly announce the traditional, **30th**—and for the second time **international**—

Rudolf Ortvay Problem Solving Contest in Physics,
from November 5, 1999, through November 15, 1999.

Every university student from any country can participate in the Ortvay Contest. PhD students compete in a separate category. The contest is for individuals; solutions sent by groups of students are not accepted. The name, the university, the major, and the university year should be given on the solutions. Pseudonyms and passwords cannot be used: each contestant has to use his/her own name.

The problems can be downloaded from the webpage of the Ortvay Contest

http://mafihe.elte.hu/ortvay

in Hungarian and English languages, in html, LaTeX and Postscript formats, from 12 o'clock (Central European Time or 11:00 GMT), Friday, November 5, 1999. The problems will be distributed by local organizers at many foreign universities, too.

Despite all the efforts of the organizers, it may happen that some unclear points or spelling mistakes stay in the text. Therefore it is very useful to visit the webpage of the contest every so often, as the corrections and/or modifications will appear there.

Each contestant can send solutions for 10 problems. For the solution of each problem 100 points can be given.

Each problem should be presented on (a) separate A4, or letter-sized sheet(s).
The contestants are kindly asked to use only one side of each sheet. Solutions written by pencil or written on thin copy-paper will not be accepted as these cannot be faxed to the referees.

Any kind of reference may be consulted; textbooks and articles of journals can be cited. Computer programs appended to the solutions should be accompanied by detailed descriptions (what computer language it was written in, how to use the program, which parameters can be set, what notations are used, how to interpret the output figures and graphs, etc.) They can be enclosed on floppy disks, or sent via email to the address given below.

Solutions can be sent by mail, fax, or email (in LaTeX, TeX or Postscript format—or, if it contains no formulae, in normal electronic mail). Please, use basic LaTeX style files. Do not use very special style files unless you include them in your file.

Postal Address: Fizikus Diákkör, Dávid Gyula,
ELTE TTK Atomfizika Tanszék,
H-1117 Budapest, Pázmány Péter sétány 1/A
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Deadline for sending the solutions: 12 o'clock CET (11:00 GMT), 15 November, 1999.

Contestants are asked to fill the form available on our webpage after posting their solutions. It will be used for identification of contestants and their solutions.

The contest will be evaluated separately for each university year, according to the total number of points. The referees reserve the right to withhold, to multiply or to share some prizes. Beyond the money prizes given for the first, second, and third places, honourable mentions and special prizes for the outstanding solutions of individual problems can be awarded. This is why it is worthwhile sending even one or two solutions.

The sponsors of this year's contest are the Roland Eötvös Physical Society, the Students' Foundation at Faculty of Sciences of Eötvös University, CAT SCIENCE BT., Tamás Serényi and Márton Major. Their help is highly acknowledged—and warmly welcome in the future.

The announcement of the results will take place on 9 December. (The place of the event is to be announced later—check it out on the website.) The detailed result will be available on the webpage of the contest thereafter. Certificates and money prizes will be sent by mail.

We plan to publish the assigned problems and their solutions in English language—to which the contribution of the most successful participants is kindly asked. The volume is planned to be distributed all over the world with the help of the International Association of Physics Students, as well as the contestant themselves. We hope this will help in making the contest even more international.

Wishing a successful contest to all our participants,

the Organizing Committee

1. Someone is swinging. She crouches when the swing is in its extremal positions and stands up when it is vertical. Examine the properties of this parametric resonance.

(Géza Tichy)

2. We shoot straight up with a good-quality slingshot. The projectile is a solid steel bullet, 1/4 inch (6.35 mm) in diameter. The projectile hits the ground nearby in 12.5 seconds. What was the initial velocity of the bullet as it left the slingshot? Do not neglect air resistance! Shooting the same bullet with the same initial velocity at an optimal angle, what is the maximum possible distance for the projectile? What is the optimal angle for this record-distance shot?

(Zsolt Frei)

3. One end of a thin rod with non-uniform mass distribution is leaning against a vertical wall, making an angle α with the horizontal plane. The other end of the rod is fixed to a pivot on the floor. Slightly displacing the rod from its unstable equilibrium position it falls down. At what height does the upper end of the rod get detached from the wall? (Friction forces can be neglected.)

(Péter Gnädig)

4. An L-shaped tube is filled with mercury. One arm of the tube is horizontal and of length l, the other one is of length h and points vertically downwards. The tube is fixed to a small carriage that can move without friction in the direction of the horizontal arm of the tube. The total mass of the carriage and the empty tube is the same as that of a mercury column of height L. Initially, the system is at rest.

   a) Give a qualitative outline of the motion of the carriage and the mercury—the latter should be referred to the glass tube.

   b) What terminal velocity does the carriage acquire if $h = l = L$?

   (The mercury flowing out of the tube does not disturb the motion of the carriage. Viscosity and surface tension can be neglected.)

(Péter Gnädig)

5. Examine the motion of a point charge in the field of a fixed electric dipole. Initially, the point charge is at rest in the symmetry plane of the dipole. After how much time, and where will it stop again? (Discuss the problem within the framework of classical mechanics. Only electrostatic forces need be considered.)

(Péter Gnädig)

6. Apart from ordinary gravity $m\mathbf{g}$, a mass point is acted on by a "drag force" $\mathbf{F}_k = -\alpha\mathbf{v}$, where the coefficient α is also velocity dependent. How should α be chosen if the *kinetic* energy of the mass point is to stay constant? What makes such a drag force look weird? Discuss ballistic motion under the above circumstances. Using this result consider a ball bouncing perfectly elastically on a slope, whose angle is $\epsilon \ll 1$, under the influence of such forces. Are there any "periodic" motions, i.e. trajectories in which the distance between subsequent bounces is a constant? Investigate the stability of such motions. What differences are there between "uphill" and "downhill" motions?

(Zoltán Kovács)

7. What trajectories $\mathbf{r}(t) = (x(t), y(t))$ can be obtained from the equations of motion derived from the following Hamiltonian?

$$H(\mathbf{p}, \mathbf{r}) = \sqrt{\frac{\mu}{m}\left(p_x^2 + p_y^2\right)\left(1 - \frac{x^2 + y^2}{a^2}\right)}$$

   (μ, m and a are positive constants).

(András Csordás)

8. Two identical metal prisms are hung up and let collide. The time of contact is measured by an electric clock. Which elastic parameter can be determined this way?

(Géza Tichy)

9. Would the length of the day increase, decrease or stay unaltered should drivers in the UK decide to keep to the right?
   Estimate the magnitude of the change (if there is any).

(Péter Gnädig—based on K. F. Riley's idea)

10. Estimate the annual change in the earth–moon distance due to tidal forces.

(Zoltán Ligeti)

11. Magnetic mixers are often used in chemistry laboratories to turn water in a glass. What is the shape of the water surface after a long period of time? The angular velocity is constant, the glass has a regular cylindrical shape.

(Imre Jánosi)

12. Find the rotationally symmetric exact solutions of the Navier–Stokes equation in two dimensions (viscous vortices), if the initial vorticity is $\omega = \delta(|\vec{r} - \vec{r}_0|)$. Construct further solutions by superposing the previous ones in the vanishing viscosity limit. What equation is satisfied by the centers of the individual vortices?

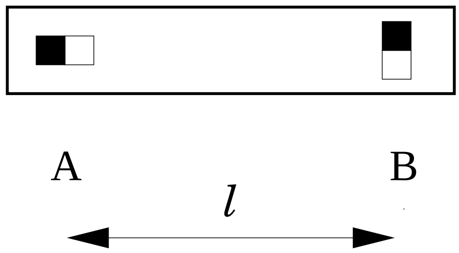
(Gyula Bene)

13. Two identical (pointlike) magnetic dipoles (A and B) arranged in the shape of a T at a distance l from each other are fixed to a ruler.

a) What is the magnitude and the direction of the torque exerted by dipole B on dipole A?

b) What is the magnitude and the direction of the torque exerted by dipole A on dipole B?

c) What happens if the ruler is hung up on a thin thread passing through its center of mass?



(The earth's magnetic field can be neglected.)

(Péter Gnädig)

14. A rather complicated circuit is made up of resistors alone. Can one determine the value of a resistance without breaking the circuit? (Voltmeters, ammeters and cells can be used.)

(Péter Gnädig)

15. A black box contains two capacitors in series charged to $+U_0$ and $-U_0$. The two outputs are connected to the positive plates of the capacitors.

a) Is it possible to determine U_0 without opening the black box?

b) Is it possible to extract the ("zero point") energy of the capacitors, or at least a part of it?

(Péter Pál Somorjai)

16. Consider an infinitely long, uniformly charged insulating rod. If it is rotated about its axis, a magnetic field is induced within the rod around the axis—whose magnitude is the weaker the farther we are from the axis—while outside the rod there is no field. Calculate the angular momentum of the electromagnetic field generated by the moving charges. Compare this contribution with the (inertial) angular momentum of the charges. Is it possible to construct a rod of negative moment of inertia? (This would be very useful, since only a cogwheel should be mounted on it, and after an initial kick it could be 'decelerated' up to its operating speed.)

(Títusz Fehér)

17. The edges of an infinite square grid are identical resistances R. The resultant resistance between two adjacent sites in the grid is known to be $R/2$.

a) How does the resultant resistance between two sites depend on the distance between them? Discuss the limit as the distance tends to infinity.

b) Consider a "perturbed" grid, which is a regular square grid with one edge (of resistance R) missing. Determine the resultant resistance between two arbitrarily chosen sites.

(József Cserti)

18. A rigid cylinder closed by a piston contains perfect gas. The gas expands adiabatically against the external pressure $p_{\rm ext}$. The initial gas pressure is $p_{\rm in}(t=0) > p_{\rm ext}$, and the initial volume is V_0. The heavy piston can move in the cylinder without friction.

a) Calculate the kinetic energy of the piston in the moment when $p_{\rm in} = p_{\rm ext}$.

b) Calculate the same quantity if the gas expands in two steps: first against an external pressure $p_{\rm x}(p_{\rm in}(t=0) > p_x > p_{\rm ext})$ up to the point $p_{\rm in} = p_{\rm x}$, and then against external pressure $p_{\rm ext}$(until $p_{\rm in} = p_{\rm ext}$).

c) The same as b) but now the gas expands in infinitely many steps. Discuss the result.

d) Suppose that a friction force acts between the wall of the cylinder and the piston through distance Δx, just before the state $p_{\rm in} = p_{\rm ext}$ is reached. While the piston decelerates, it loses all its kinetic energy and stops exactly when $p_{\rm in} = p_{\rm ext}$.
The heat produced by friction is absorbed totally by the environment of temperature T through a slow process of temperature equalization.

When is the total entropy increase of the system and the environment largest, if a larger friction force acts through a shorter distance Δx, or if a smaller friction force acts through a longer distance Δx?

(Péter Hantz)

19. A flat black disk is floating in space. Heat conduction between its two sides is negligible. The disk is moving (without rotating) parallelly to its own axis at a (small) velocity v_0 with respect to the cosmic background radiation. Initially the temperatures of both sides are equal to that of the CBR. How will its velocity change in time on interacting with the CBR? Can it change sign? If so, what material is the disk made of?

(Títusz Fehér)

20. If a coin is tossed, the outcome will be randomly heads or tails—at least to a good approximation. The reason for this is the sensitive dependence of the motion of the spinning coin to the initial conditions. One can, however, examine the problem the following way. A quarter (500-lira/5-mark ... coin) with its tails up is pushed slowly off a table and the outcome (heads/tails) is recorded—and the same is repeated many times. Show that the longer the time of falling the more even the heads/tails distribution. Define the phase space of the spinning coin and estimate the Lyapunov exponent of the motion from the experimental results.

(Péter Pollner)

21. The day-by-day variation of the price of a given stock can be described as a random walk. Initially, each broker has 100 units worth of stocks. Then they get involved in the following sequence of transactions. Participants have to sell all their stocks in a random moment of the interval $[0, \theta]$ (where $\theta \gg 1$ day is a fixed constant). After selling their stocks, they get a 24-hour rest, and then a new interval of length θ starts, in a randomly chosen moment of which they have to buy new stocks for all their money. After buying the new stocks they take another 24-hour rest, and the interval of length θ set for selling starts again, and so forth. Transaction costs are assumed to be zero.

a) Find the probability distribution (PD) of the expected profit after a long period of time. Does this PD depend on θ?

b) What is the probability distribution of the highest profit (i.e. the value of stocks of the richest broker in a fixed moment after a long period of time)?

(Imre Jánosi)

22. The dynamics of a complex system (such as a macromolecule that has states of complicated conformation) is described by transitions between metastable states. For each metastable state α ($\alpha = 1, \dots M, M \gg 1$) there exists a lifetime τ_a, which describes the exponentially decaying escape probability from state α, i.e. denoting the probability of escaping from state α between times t and $t + dt$ by $p_\alpha(t)dt$, the functional relation $p_\alpha(t) \sim \exp(-t/\tau_\alpha)$ holds. Let the number of metastable states between times τ and $\tau + d\tau$ be denoted by $M\psi(\tau)d\tau$, where

$$\psi(\tau) = \begin{cases} 0, & \text{ha } \tau \ll \tau_0 \\ \sim \tau^{-1-x}, & \text{ha } \tau > \tau_0. \end{cases}$$

x is usually a temperature-dependent parameter of the system for which $0 < x < 1$. At $t = 0$ the system chooses randomly among the M states, and likewise, after escaping from a trap α it will fall into a randomly chosen trap.

a) Examine the system for time $t_w \gg \tau_0$, and estimate the longest interval of time the system spent in one metastable state.

b) Construct a "master" equation for $P_\alpha(t)$ (where $P_\alpha(t)$ gives the probability that the system is in state α at time t).

c) Introduce the free energy f_α of the metastable states in such a way that the Gibbs–Boltzmann formula

$$P_\alpha^{\mathrm{eq}} \propto \exp(-f_\alpha/kT)$$

holds for the equilibrium distribution P_α^{eq}. What is the connection between f_α and τ_α?

d) Calculate the probability $\Pi(t_w, t)$ that between times t_w and $t_w + t$ the system does *not* make a transition from one state to another one.

(Tamás Temesvári)

23. In the usual notation, the scattering of a particle of momentum $\mathbf{k}$ by a potential $U(\mathbf{r})$ is constructed via solving the Schrödinger equation

$$\left[\nabla^2 + k^2\right]\psi(\mathbf{r}) = U(\mathbf{r})\psi(\mathbf{r}).$$

It is usually stated—without preamble—that the asymptotic solution to this equation *has/must have* the form

$$\psi(\mathbf{r}) = \psi_{\mathbf{k}}(\mathbf{r}) \sim \exp(i\mathbf{k}\mathbf{r}) + \frac{\exp(ikr)}{r} f_{\mathbf{k}}(\mathbf{n}),$$

where $k = |\mathbf{k}|$, $r = |\mathbf{r}|$, $\mathbf{n} = \mathbf{r}/r$, and $f_{\mathbf{k}}(\mathbf{n})$ is the so-called scattering amplitude. It is rarely mentioned, however, that this expression is obtained by converting the preceding one into an integral equation, and expanding—for short range scatterers—therein the wave function in an asymptotic series in powers of $1/r$, and keeping only the first term in the expansion.

a.) Take $U(\mathbf{r})$ to be not too singular at the origin, and vanishing as $r^p U(\mathbf{r}) \to 0$ for all p as r goes to infinity. Derive the *complete* asymptotic series of which the second term on the right hand side of the previous expression is the first term.
Hint: Do not use spherical harmonics.

b.) What new information is contained in the higher order terms? What is the physical significance of these terms?

(Péter Magyar—based on M. L.'s idea)

24. Stop the runaway electron! For what real constants a, b, c does the wave function

$$\Psi(x,t) = Ai\left[\frac{x + at^2}{x_0}\right]\exp(ibxt + ict^3)$$

satisfy the one dimensional Schrödinger equation of a *free particle*? The parameter x_0 is real, and Ai is the Airy function defined as

$$Ai(x) = \int_0^\infty \cos(w^3/3 + wx)dw.$$

How does this wave packet move? What is its acceleration? Since a free particle is being discussed, no force acts on it. How come it is accelerated?
Solve the Schrödinger equation in the linear, time-dependent potential $V(x,t) = g(t)x$. To this end, reparametrize space and the wave function according to

$$\begin{aligned} x' &= x + \alpha(t), \\ \Psi(x,t) &= e^{i\Phi(x,t)}\Psi'(x',t). \end{aligned}$$

In this case Ψ' describes a particle moving in the potential $V'(x',t)$ (for a suitably chosen Ψ.) Choosing such an α that $V'(x',t)$ is independent of x', the problem is traced back to that of a free particle, which has been solved above. Thus we also have the wave function of the particle moving in potential V, which is also an Airy function—write it up, please! How should $g(t)$ be chosen so that the wave does *not move*?

What can be said about the effects of changing over to accelerated coordinate systems on wave functions?

(Gábor Veres)

25. Let S^3 denote the 3-dimensional spherical surface in 4-dimensional space, i.e. $S^3 = \{\sum_{i=0}^3 u_i^2 = R^2\}$. The upper half of S^3 (i.e. points of S^3 with $u_0 > 0$) is chosen as the configuration space (CS) of a mechanical system. Each point of the CS is projected onto the (3D) space tangential to the sphere in its north pole $u_0 = R$, by lines passing through the origin ($u_i = 0; \forall i$). Let the image of point (u_1, u_2, u_3) be denoted by (x, y, z).

Write down the Lagrangian of free motion as a function of the coordinates of the tangent space. Introducing the shorthand $r = \sqrt{x^2 + y^2 + z^2}$, study the motion in the Kepler potential $V(r) = \alpha/r$. Determine all independent conserved quantities and calculate their Poisson brackets.
Use the above result to find the spectrum of the Hamiltonian operator of the quantum case.

(Zoltán Bajnok)

26. Consider the generalization of a simple pendulum of mass m and length l in uniform gravity field g, in which the mass point reaches its initial position after $k \geq 1$ revolutions (instead of 1).

Classically, this system has k equivalent lowest-energy states. What can be stated about the k lowest-lying states and their energies after quantization?

Study the discrete symmetries of the system both classically and quantum mechanically, including tunneling effects.

(Zoltán Bajnok)

27. When a function is represented on a lattice, the second derivative is customarily approximated by

$$\left.\frac{d^2 f}{dx^2}\right|_i = \frac{f_{i-1} - 2f_i + f_{i+1}}{\Delta x^2},$$

where $f(i)$ is the value of the function in the ith lattice point, and Δx is the lattice spacing.
Consider a cubic lattice and a cube in it centered on the origin, whose vertices are lattice sites and whose edges are parallel to the axes x, y, z of the lattice.

Take a deep breath, and define the approximation of the second derivative with respect to angle ϕ about the z-axis. Define the approximating operator L_z^2 and write it down in matrix notation. Define also the approximating operators L_x^2, L_y^2, $\mathbf{L}^2$ and give their matrix representations.

Next, solve the eigenvalue problem of the matrices. How well do these approximate the spectrum of the 'real' operators? What are the eigenfunctions (eigenvectors)? How are they related to the 'real' ones? Is there a way to define the approximating operators L_x, L_y, L_z? How? What about their commutation relations?

Extra question: Is it just a coincidence that the spectrum of the approximating operator $\mathbf{L^2}$ is practically the same as that of three independent spins 1/2 placed in a magnetic field?

(Zsolt Bihary)

28. Model the electronic structure of a diatomic molecule as follows. Each atom has two electrons that are identical except for their spin. Using second-quantized formulation, the Hamiltonian in this base is given as

$$\begin{aligned}
H &= H_0 + H_t + H_e \\
H_0 &= -\varepsilon_1 \left(a_1^+ a_1 + b_1^+ b_1\right) - \varepsilon_2 \left(a_2^+ a_2 + b_2^+ b_2\right) \\
H_t &= -t \left(a_1^+ a_2 + a_2^+ a_1 + b_1^+ b_2 + b_2^+ b_1\right) \\
H_e &= U \left(a_1^+ a_1 b_1^+ b_1 + a_2^+ a_2 b_2^+ b_2\right),
\end{aligned}$$

where a_i and b_i are the ith annihilation operators for atomic orbitals having positive and negative spins, respectively. ε_1 and ε_2 are the respective energies of the orbitals, and t is the hopping parameter between the orbitals. U is the strength of the repulsive interaction between electrons of the same atom.

a) What is the dimension of the Fock space?

b) Prove that the operator of the total number of electrons commutes with the Hamiltonian. Use this to write down in explicit matrix form the blocks of H belonging to different total electron numbers. How many blocks are there and what are their respective dimensions?

c) Prove that the operator of the total spin also commutes with H. This can be useful in finding the answers to the following questions.

d) What is the ground-state energy in the case of 1, 2, 3 and 4 electrons?

e) How do the chemical properties of the molecule depend on the parameters of H in the 2-electron case? Is the molecule non-polar, slightly polar or polar in its ground state? Will the molecule dissociate into an atomic or an ionic state if the atoms are separated slowly (i.e. the value of t approaches 0)? Interpret the results in the language of chemistry, too.

(Zsolt Bihary)

29. An atom in an excited state traverses an empty Fabry–Perot interferometer sideways. The excitation energy coincides with an eigenfrequency of the interferometer. Calculate the motion of the atom in semiclassical approximation. What state is the system in after the atom has traversed?

(Gyula Bene)

30. Traditionally, quantum measurements have been associated with irreversible transformations of *pure states* into *mixed states* (see, e.g. von Neumann's postulates about the density operator $\hat{\rho}$).

Now consider the—vastly oversimplified—case where the dynamical variable $\hat{s}_z$ of an electron is measured by coupling it to a "measuring apparatus" consisting of another spin-1/2 particle, operating according to the following principle,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ \beta \end{pmatrix},$$

the first (second) term referring to the coordinates of the apparatus (electron), respectively.

a) Construct the interaction Hamiltonian $\hat{H}_i$ corresponding to the above measuring process.

b) Write down the matrices corresponding to the spin components of the electron, and express the density matrix $\hat{\rho}$ for the coupled system in terms of α and β. Use a suitable definition, and answer the question: all things considered, is this measurement reversible or not?

Hint: forget about textbook dogmas, and work within the framework of this model.

(Magyar Péter—based on A. P.'s idea)

31. Carbon nanotubes have attracted a lot of attention since their discovery in 1991. A straight carbon nanotube can be regarded as a graphite sheet rolled up in a cylindrical form. Typical cylinder diameters are in the nanometer range.

Among the several possible constructions of a nanotube, consider the "armchair" fiber that is obtained by rolling up the graphite sheet in such a way that every third edge of the hexagons is perpendicular to the cylinder axis.

Applying a uniform magnetic field parallel to the cylinder axis a gap appears in the electronic band structure at the Fermi energy. Find the magnetic field dependence of the width of the gap.

Can you account for a magnetic-field-induced metal–insulator transition?

It is sufficient to consider only effects of π orbitals in the tight-binding model.

(József Cserti)

32. A particle oscillates in a countably infinite dimensional anisotropic harmonic potential. The dimensions are labelled by non-negative integers, $0, 1, \ldots$.

In the 0th dimension, the potential is extremely flat, while in the other directions the angular frequency ω is proportional to

a) the index of the dimension

b) the square of the index

The system is then perturbed by a spherically symmetrical, relatively flat potential. Carry out a quantum mechanical calculation on the effective equation of motion in the 0th dimension. What kind of difficulties arise? What is the model similar to?

(Szabolcs Borsányi)

33. Egon Quark, having got tired of the everlasting commutation between Philadelphia and Budapest, decided to build a modest tunnel through the homogeneous earth, starting in Budapest and ending in Philly. Egon would get in his underground sleigh in Budapest, set off with 0 initial speed, and some time later would get out in Philadelphia. However, he is usually short of time—being a physicist—so he calculated the optimal shape of the tunnel.

Believe it or not, he got the financial support, but some problems emerged: the construction company lost Egon's plans, so the physicist-in-chief made a simplified last–minute calculation. He drew a circle, put Philadelphia (P) and Budapest (B) on it—symmetrically? Don't be so naive! Then he calculated the brachistochrone curve, as if a uniform gravity field were present, and drew it in the inside of the circle. To avoid further approximation, he calculated the real transit time inside the earth, i.e. he dropped the approximation of a uniform gravity field at this point.

Finally, he tried to make up for his mischief by choosing points P and B as best as he could.

How much time will Egon lose this way?

(Attila Piróth)

34. When people work, they get tired, and this affects their efficiency, too. Let us model this the following way:

$$Q(t) = P(t)\,\varepsilon(t),$$

where $P(t)$ is the instantaneous performance, $\varepsilon(t)$ is the instantaneous efficiency, and $Q(t)$ is the instantaneous effective performance.

$$\varepsilon(t) = 1 - f(t),$$

where $f(t)$ is the instantaneous tiredness, which makes our efficiency smaller. If someone is very tired, the efficiency might even be negative, since a zombie rather destroys than does useful work. Tiredness is the cumulative function of the previous performance:

$$f(t) = \frac{1}{P_0\tau}\int_{-\infty}^{t} dt' P(t') \exp((t'-t)/\tau),$$

where P_0 and τ are human constants characteristic of the person working.

a) Assume one works at a constant performance. How does one's effective performance change with time? What is one's effective performance after a good while? At what performance value can it be maximized? What is the efficiency in this case? How is this affected by being tired/restful initially? What do the human constants mean?

b) Next, assume that our boss expects us to work with a constant effective performance. How should we change our performance to come up to this expectation? For how long can we persist? What are the human constants preferred by the boss? What happens if someone starts working with a hangover?

c) Suppose a certain amount of work has to be completed in a certain amount of time. What is the best working strategy if we want to complete the task with investing the least possible performance?

(Zsolt Bihary)

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